**ENEE324: Engineering Probability**

**Final Examination** **May 19, 2015**

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**Max total 60 pts.** Instructor: A. Barg

• Please be precise and rigorous in your statemens, and show all your calculations. Be sure to **Print Your Name**

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**Problem 1** (10pts) Consider a Markov chain with the transition matrix

$$P = \begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{pmatrix}, \quad 0 \le p \le 1/2.$$

(a) Classify the states into absorbing, recurrent, transient and identify recurrent classes. Make sure to consider the boundary cases $p = 0, p = 1/2$.

(b) Compute the steady-state distribution of the chain. Pay attention to the boundary cases.

(c) Compute the probabilities $r_{ij}(2)$ of transitioning from $i$ to $j$ in 2 steps for all $i, j = 1, 2, 3$.

*Solution:* (a) If $p = 0$ then all the states are absorbing, and the chain has 3 "recurrent classes" formed by the states. If $0 < p \le 1/2$, then all the states are recurrent, and the chain has a single non-periodic recurrent class.

(b) If $p = 0$, the chain has 3 steady-state distributions depending on the state $X_0$, which are $(1,0,0), (0,1,0), (0,0,1)$. If $0 < p \le 1/2$, then $\pi = \pi P$ gives the equations

$$\begin{aligned}
\pi_1 &= \pi_1(1-2p) + \pi_2 p \\
\pi_2 &= \pi_1 \cdot 2p + \pi_2(1-2p) + \pi_3 \cdot 2p \\
\pi_3 &= \pi_2 p + \pi_3(1-2p)
\end{aligned}$$

which yield $\pi_1 = \pi_3 = 1/4, \pi_2 = 1/2$.

(c) Computing $P^2$, we obtain

$$R(2) = \begin{pmatrix} (1-2p)^2 + 2p^2 & 4p(1-2p) & 2p^2 \\ 2p(1-2p) & 4p^2 + (1-2p)^2 & 2p(1-2p) \\ 2p^2 & 4p(1-2p) & 2p^2 + (1-2p)^2 \end{pmatrix}.$$

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**Problem 2** (10pts) The joint density of RVs X and Y is as follows:

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0 \le x < \infty, 0 \le y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of the random variable X/Y.

*Solution:*
Compute the CDF of $Z = X/Y$. For $z > 0$ ,

$$F_Z(z) = P\Big(\frac{X}{Y} \le z\Big) = \iint_{X/Y \le z} e^{-(x+y)}\, dx\, dy =$$
$$= \int_{y=0}^{\infty} \int_{x=0}^{zy} e^{-(x+y)}\, dx\, dy$$
$$= \int_{y=0}^{\infty} (1 - e^{-zy}) e^{-y}\, dy$$
$$= \int_{y=0}^{\infty} (1 - e^{-zy}) e^{-y}\, dy$$
$$= \Big\{ (-e^{-y} + \frac{e^{-y(z+1)}}{z+1}) \Big\} \Big|_0^{\infty} = 1 - \frac{1}{z+1}$$

Now find

$$f_{X/Y}(z) = \frac{d}{dz} F_Z(z) = \frac{1}{(z+1)^2}, \quad 0 < z < \infty.$$

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**Problem 3** (10 pts) In the tropical forest of Luamo island, thunderstorms occur all year round. They happen at a Poisson rate of 5 per month.

(a) What is the probability that in a given calendar year there are a total of two thunderstorms in January and August combined?

(b) What is the probability that in a given calendar year there are exactly two (not necessarily consecutive) months out of the twelve months that see no thunderstorms at all?

*Solution:* For the Poisson process, let $P(k, t)$ be the (Poisson) probability of $k$ arrivals in time $t$. Then

$$p_0 \triangleq P(0,1) = e^{-5}, p_1 \triangleq P(1,1) = 5e^{-5}, p_2 \triangleq P(2,1) = \frac{25}{2} e^{-5}.$$

(a) The probability of the event of interest is

$$2p_0 p_2 + p_1^2 = 50e^{-10} \qquad \text{OR:} \quad P(2,2) = e^{-2\times 5} \frac{(2\times 5)^2}{2!} = 50e^{-10}.$$

(b) $P(\text{two dry months}) = \binom{12}{2} e^{-10} (1 - e^{-5})^{10}$.

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**Problem 4** (10pts)
Two students are to meet in the Student Union. If each of them independently arrives at a time uniformly distributed between 12 noon and 1 pm, find the probability that the first to arrive has to wait longer than 10 minutes.

*Solution:*
Place the origin at 12 noon, and let $X \sim Unif[0,1], Y \sim Unif[0,1]$. The event of interest is that $\{|X - Y| > 1/6\} = \{Y < X - 1/6\} \cup \{Y > X + 1/6\}$. Since $P(\{Y < X - 1/6\})$ is the area of the isosceles right triangle with legs $5/6$, i.e., $25/72$, the answer is $2 \times 25/72 = \mathbf{25/36}$.

Or, if you missed the geometric view, compute the integrals:

$$2P[X + 10 < Y] = 2 \iint_{x+10<y} f(x,y)\, dx\, dy$$

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\begin{aligned}
&= 2 \iint_{x+10<y} f_X(x) f_Y(y)\, dx\, dy \\
&= \int_{y=10}^{60} \int_{x=0}^{y-10} (\frac{1}{60})^2\, dx\, dy \\
&= \frac{2}{(60)^2} \int_{y=10}^{60} (y-10)\, dy = \frac{25}{36}.
\end{aligned}
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**Problem 5** (10pts) The distance between the towns T1 and T2 is 11 miles, and there are 10 mile markers on the road from T1 to T2, with readings 1,2,...,10. A marker is chosen randomly with a uniform distribution. Let $D_1$ be the distance from T1 to the chosen marker, and let $D_2$ be the distance from $T2$ to that same marker.

(a) Are $D_1$ and $D_2$ positively or negatively correlated? Justify your answer.

(b) Compute the expected value of $D_1 \times D_2$.

*Solution:* (a) If $D_1$ increases, then $D_2$ decreases, so they are negatively correlated. To justify this formally, write

$$\rho_{D_1 D_2} = \frac{E[(D_1 - ED_1)(D_2 - ED_2)]}{\sigma_{D_1}\sigma_{D_2}}$$

and notice that if $ED_1 = ED_2 = 5.5$, and the quantities $(D_1 - ED_1)$ and $(D_2 - ED_2)$ always have opposite signs. Therefore, $\rho_{D_1 D_2} < 0$.

(b) Let $X$ be a uniform RV with pmf $p_X(k) = 0.1, k = 1, \dots, 10$ and let $Z = X(11 - X)$. The pmf of $Z$ is

$$p_Z(k) = 0.2,\ k = 10, 18, 24, 28, 30.$$

Then $EZ = 0.2 \times 110 = 22$.

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**Problem 6** (10pts) Suppose that $n$ cars passed through a certain intersection within an hour, and $k$ out of them were passenger cars. The probability that any given car is a passenger car is $p$ and the type of each car is independent of the other cars. Let $i$ be a number between 1 and $n$. What is the probability that the $i$th car was a passenger car?

*Solution:* Let $B$ be the event that the $i$th car is a passenger car and let $A$ be the event that $k$ out of $n$ cars are passenger. Then

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{p\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.$$

The answer does not depend on $p$.