ENEE324: Engineering Probability: Final Examination. May 14, 2016

There are 6 problems, 10 points each. **MAX 60** Instructor: A. Barg

- **Print Your First and Last Name!**
- **Clearly identify your answers to the questions!** No credit for an answer with no justification.

Problem 1. We run a sequence of experiments each of which can result in either success or failure. Two consecutive successes are followed by a success with probability 0.8; in all other cases the next experiment is a success with probability 0.5. In the long run, which proportion of the experiments ends in success?

(Begin with setting up a Markov chain and finding its stationary distribution. After that, argue which proportion of the outcomes are a success by conditioning the current outcome on the two previous outcomes.)

Solution 1: Let '1' denote a success and '0' denote a failure. Let $S_j, j = 1, 2, 3, 4$ be the states of the chain, where $S_1 = 00, S_2 = 01, S_3 = 10, S_4 = 11$. The matrix of transition probabilities is

	00	01	10	11
00	0.5	0.5	0	0
01	0	0	0.5	0.5
10	0.5	0.5	0	0
11	0	0	0.2	0.8

For the stationary distribution we obtain the equations

$$\pi_1 = 0.5\pi_1 + 0.5\pi_3,\ \pi_2 = 0.5\pi_1 + 0.5\pi_3,\ \pi_3 = 0.5\pi_2 + 0.2\pi_4,\ \pi_4 = 0.5\pi_2 + 0.8\pi_4$$

whence using $\sum_j \pi_j = 1$, we get $\pi_1 = \pi_2 = \pi_3 = 2/11; \pi_4 = 5/11$. Next let X_i be the RV that denotes the ith state of the chain. Using the total probability formula, we obtain for $i \geq 2$

$$P(X_i = 1) = \sum_{j=1}^{4} P\{X_i = 1 | (X_{i-2}X_{i-1}) = S_j\} \cdot P\{(X_{i-2}X_{i-1}) = S_j\}.$$

In the long run, $P\{(X_{i-2}X_{i-1}) = S_j\} = \pi_j, j = 1, \dots, 4$ so

$$P(X_i = 1) = \pi_1 p_{12} + \pi_2 p_{24} + \pi_3 p_{32} + \pi_4 p_{44} = \frac{2}{11}\frac{3}{2} + \frac{5}{11}\frac{8}{10} = \frac{7}{11} \approx 0.64.$$

Solution 2: Make a chain of the following 3 states:

$$S_0 : X_i = *0; \quad S_1 : X_{i-1} = 0, X_i = 1; \quad S_2 : X_{i-1} = X_i = 1$$

Then the transitions are governed by the following matrix:

	S_0	S_1	S_2
S_0	0.5	0.5	0
S_1	0.5	0	0.5
S_2	0.2	0	0.8

The stationary distribution is $\pi_0 = 4/11; \pi_1 = 2/11; \pi_2 = 5/11$ (we lump together states S_1 and S_3 from Solution 1, making a new state S_0).

The proportion of time the chain is in states 1 or 2 equals $1 - \pi_0 = \frac{7}{11}$.

Problem 2. Let X be a discrete RV taking integer values. The CDF of X is given by

$$F_X(x) = \begin{cases} 0, & x < -2 \\ 1/3, & -2 \leq x < 0 \\ 3/4, & 0 \leq x < 3 \\ 1, & x \geq 3. \end{cases}$$

Find $EX, E(|X|^2), E(X \cdot |X|), E[(X - 2|X|)^2]$ (Hint: begin with finding the PMF of X).

Solution: We have

$$p_X(-2) = 1/3; p_X(0) = 3/4 - 1/3 = 5/12; p_X(3) = 1/4$$
$$p_{|X|}(2) = 1/3; p_{|X|}(0) = 5/12; p_{|X|}(3) = 1/4$$

Then

$$EX = -2/3 + 3/4 = 1/12$$
$$E(|X|^2) = 4/3 + 9/4 = 43/12.$$

Let $Z = X \cdot |X|$, then Z takes the values $-4, 0, 9$, and the pmf of Z is

Z	-4	0	9
$p_Z(z)$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{4}$

Then

$$EZ = -4/3 + 9/4 = 11/12.$$

Finally $EX^2 = E|X|^2$, so

$$E[(X - 2|X|)^2] = EX^2 + 4E|X|^2 - 4E(Z) = \frac{5 \cdot 43}{12} - \frac{44}{12} = \frac{57}{4} = 14\frac{1}{4}.$$

Answer: 1/12; 43/12; $-7/12$; 20.25.

Problem 3. (a) We select 100 independent random real numbers from the set $\{x : 0 \le x \le m\}$, where m is a positive integer. These numbers are then rounded down to 3 decimal points. What is the probability that at least 2 of the obtained results equal 0.981?

(b) A Poisson process registers arrivals with rate λ per hour. In the span of 20 hours, what is the probability that there are three non-overlapping one-hour intervals (each starting on the whole hour; not necessarily consecutive) with exactly two arrivals within each of them?

Solution: (a) Let $p = 0.001/m$, then the probability that each of the numbers falls in the interval $[0.981, 0.982)$ is p, and these events are independent. Therefore, using the binomial distribution, the probability that at least 2 numbers are within this interval equals

$$\text{Answer:} \qquad 1 - \text{Binom}(100, p, 0) - \text{Binom(100,p,1)} = 1 - (1-p)^{100} - 100p(1-p)^{99}.$$

Or, we can use the Poisson approximation: setting $\lambda = np = 1000m$, write

$$\Pr(\ge 2 \text{ numbers}) \approx 1 - e^{-\lambda}(1 + \lambda).$$

(b) Let p be the probability that there is exactly one arrival within a one-hour period. Then $p = P(2,1) = \frac{\lambda^2}{2}e^{-\lambda}$. Now the answer is given by a binomial probability Binom$(20, p, 3)$.

Answer: $\binom{20}{3}p^3(1-p)^{17} = 1140p^3(1-p)^{17}$.

Problem 4. Let X and Y have the joint PDF given by

$$f_{XY}(x, y) = 1 \text{ if } 0 \le x \le 1, 0 \le y \le 1$$

and 0 otherwise.

Find $P(X + Y \le 1/2), P(X - Y \le 1/2), P(XY \le 1/4), P(X^2 + Y^2 \le 1)$.

Solution: By computing the areas, we obtain

$$P(X + Y \le 1/2) = 1/8;$$
$$P(X - Y \le 1/2) = 7/8,$$
$$P(XY \le 1/4) = \frac{1}{4} + \int_{1/4}^{1} \frac{dx}{4x} = \frac{1}{4}(1 + \ln 4).$$
$$P(X^2 + Y^2 \le 1) = \pi/4.$$

If you prefer to compute the integrals, then the following is a correct way to write them:

$$P(X+Y\le 1/2)=\int_0^{1/2}\int_0^{\frac12-y}dxdy=\frac12(y-y^2)\Big|_{1/2}^{0}=\frac18.$$

$$P(X-Y\le 1/2)=\int_0^1\int_{\max\{0,x-\frac12\}}^1 dydx=\frac12+\int_{1/2}^1\int_{x-\frac12}^1 dydx=\frac12+\Big[\frac12+x\Big]_{1/2}^1=\frac78$$

$$P(XY\le 1/4)=\int_0^1\int_0^{\min\{1,\frac{1}{4y}\}}dxdy=\int_0^{1/4}\int_0^1 dxdy+\int_{1/4}^1\int_0^{\frac{1}{4y}}dxdy=\frac14+\int_{1/4}^1\frac{dy}{4y}dxdy$$
$$=\frac14+\frac14\ln y\Big|_{1/4}^1=\frac14(1+\ln 4).$$

$$P(X^2+Y^2\le 1)=\int_0^1\int_0^{\sqrt{1-y^2}}dxdy=\int_0^1\sqrt{1-y^2}dy$$
$$=\int_0^{\pi/2}\cos^2 tdt=\int_0^{\pi/2}\Big(\frac12+\frac12\cos(2t)\Big)dt=\frac{\pi}{4}+\frac14\sin(2t)\Big|_0^{\pi/2}=\frac{\pi}{4}.$$

Problem 5. We are given n independent random samples $X_1,\dots,X_n$ from an unknown distribution of an RV X with variance $\mathrm{Var}(X)=3$.

(a) Let $\bar X_n=\frac1n\sum_{i=1}^n X_i$. Find $E\bar X_n,\mathrm{Var}(\bar X_n)$.

(b) We guess that the true value of EX equals $\bar X_n$. Suppose that our goal is that this guess differs from the true value of EX by at most 4. What is the probability that the estimate $\bar X_n$ achieves this precision if $n=5$?

Solution: (this is a rephrased Prob. 4 from HW8) Consider the RV $\bar X_n$, then $E\bar X_n=EX,\mathrm{Var}(\bar X_n)=\frac3n$. Using the Chebyshev inequality, $P(|\bar X_n-EX|\ge 4)\le\frac{3}{5\cdot 4^2}=0.0375$. The event in question is complementary, so we find the answer to be $1-0.0375=0.9625$.

Problem 6. (a) Given n balls, we place them randomly, one by one, into Urn 1 (with probability p) or Urn 2 (with probability $1-p$). Let X_1 be the number of balls in Urn 1 at the end of the experiment, and let $X_2=n-X_1$ be the number of balls in Urn 2. What is the pmf and the mean of $X_i, i=1,2$? Are X_1 and X_2 independent?

(b) Now assume that the number of balls n is a Poisson RV with parameter λ. Once we have a realization of this RV, i.e., the value of n, we perform the same experiment as in part (a), obtaining the RVs X_1 and X_2. These RVs are different from those in part (a) because they depend on the realization of n. Find the PMFs and the means of X_1 and X_2 by conditioning on n and using the total probability theorem.

Solution: (a) $X_1\sim\mathrm{Binom}(n,p), EX_1=np;\ X_2\sim\mathrm{Binom}(n,1-p), EX_2=n(1-p)$. Given the value of X_1 we know exactly the value of X_2, i.e., $P(X_2=j)\ne P(X_2=j|X_1=k)$, so X_1 and X_2 are dependent.

(b) Using the total probability formula,

$$P(X_1=i)=\sum_{n=0}^\infty P(X_1=i|n)P(n)=\sum_{n=i}^\infty\binom{n}{i}p^i(1-p)^{n-i}e^{-\lambda}\frac{\lambda^n}{n!}$$
$$=\Big(\frac{p}{1-p}\Big)^i e^{-\lambda}\sum_{n=i}^\infty\frac{n!}{i!(n-i)!}\frac{((1-p)\lambda)^n}{n!}$$
$$=\Big(\frac{p}{1-p}\Big)^i e^{-\lambda}\frac{((1-p)\lambda)^i}{i!}\sum_{n=i}^\infty\frac{((1-p)\lambda)^{n-i}}{(n-i)!}=\frac{(p\lambda)^i}{i!}e^{-\lambda}e^{(1-p)\lambda}$$
$$=\frac{(p\lambda)^i}{i!}e^{-p\lambda},\quad i=0,1,\dots.$$

Answer: $X_1\sim\mathrm{Poisson}(p\lambda), EX_1=p\lambda$ and similarly $X_2\sim\mathrm{Poisson}((1-p)\lambda), EX_2=(1-p)\lambda$.

Or, you can argue that we are splitting the Poisson process for n into two streams, obtaining two Poisson processes with rates $p\lambda$ and $(1-p)\lambda$.

(It is an interesting question to figure out whether in this case X_1 and X_2 are independent, by computing $P(X_1 = i | X_2 = k)$.)