ENEE324: Engineering Probability

Final Examination **May 20, 2019**

6 Problems, Each problem is worth 10 points; Max 60 points

Instructor: A. Barg

• Please be precise and rigorous in your statements, and show all your calculations. Please write neatly and legibly. Be sure to **Print Your Name!**

• The problem statements are printed on both sides of this sheet.

• This is a closed book exam, but you are allowed up to two 8.5× 11 pages of notes. No calculators please! Good luck!

Problem 1. The number of traffic accidents on any given day is a Poisson random variable with mean 2, and these random variables for different days are independent.

(a) What is the probability that there are $i = 0, 1, 2, \dots$ accidents on a given day? (give the answer for each positive integer value of i)

(b) What is the probability that the number of accidents on a given day is different from two (i.e., equal to either zero or one or three, etc.)?

(c) Take some 5 days, for instance, Mo-Fri this week. What is the probability that at least three of these five days each have exactly two accidents?

(d) Take some 2 days. What is the probability that there are a total of six accidents over these two days? (for full credit your answer should not include any sums)

(e) If each accident is, independently, a "major accident" with probability p, what is the probability there are no major accidents on a given day? (for full credit your answer should not include any sums).

SOLUTION (a) $P(i) = e^{-2}2^i/i!, i = 0, 1, 2, \dots$.

(b) $1 - 2e^{-2}$.

(c) Let $p = 2e^{-2}$ to be the probability of 2 accidents in a day, then the answer is given by the binomial probability $10p^3(1-p)^2 + 5p^4(1-p) + p^5$.

(d) We are computing the probability that $P(N(2) = 6) = \frac{(2\lambda)^6}{6!}e^{-2\lambda} = \frac{256e^{-4}}{45}$.

(e) $\sum_{i=0}^{\infty} e^{-2}2^i(1-p)^i/i! = e^{-2+2(1-p)} = e^{-2p}$.

OR: The split Poisson process of "no major accidents" has arrival rate $2p$, so $P(0,1) = e^{-2p}$.

Problem 2.

(a) Compute the moment generating function $g_X(s)$ of the exponential random variable X with parameter λ.

(b) Use the expression for $g_X(s)$ from Part (a) to compute the nth moment $E(X^n)$ for all positive integer values of $n = 1, 2, \dots$. Using this calculation, find the variance of X (to receive credit for this problem, do not use other ways of computing the moments and the variance of X).

SOLUTION (a) $g_X(s) = \int_0^{\infty} e^{sx-\lambda x}dx = \frac{\lambda}{\lambda - s}, s < \lambda$.

(b) $EX^n = \frac{d^n}{dx^n} g_X(0) = \frac{\lambda n!}{(\lambda-s)^{n+1}}\Big|_{s=0} = \frac{n!}{\lambda^n}$. Thus, $EX = \frac{1}{\lambda}, EX^2 = \frac{2}{\lambda^2}$ and $\mathrm{Var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$.

Problem 3. The repair time of a device is an exponential random variable X with mean 1/2 day.

(a) Let A denote the event that the repair time is greater than 1 day. Find $P(A)$ (your answer should not include any integrals).

(b) Compute the conditional PDF of X given the event A defined above. Using the conditional PDF, find the expected time of repair conditional on A (for full credit, your answer should be a single number).

(c) There is a way to obtain the answer in Part (b) without any computations. Identify this way.

SOLUTION (a) Let A denote the event $\{X > 1\}$. We have $P(A) = \int_1^\infty 2e^{-2x}dx = 1 - F_X(1) = e^{-2}$.

(b) The conditional PDF is $f_X(x|A) = \frac{f_X(x)}{P(A)} = e^2(2e^{-2x}) = 2e^{2(1-x)}$. Then

$$E(X|A) = \int_1^\infty x2e^{2-2x}dx = 1.5$$

(c) Since X is memoryless, $P(X > 1+x|X > 1) = P(X > x)$. Thus, the conditional expectation $E(X|A)$ is $1 + EX = 1 + 1/2 = 1.5$.

Problem 4. The joint CDF of the lifetimes of two light bulbs is

$$F_{XY}(x, y) = (1 - e^{-x^2})(1 - e^{-y^2}) \text{ if } x > 0, y > 0, \quad \text{and } F_{XY}(x, y) = 0 \text{ if either } x < 0 \text{ or } y < 0.$$

(a) Find the probability that the first light bulb lasts at least twice as long as the second (you will start with finding the joint PDF).

(to receive full credit your answer should be a single number)

(b) Find the probability that one of the light bulbs lasts at least twice as long as the other.

(to receive full credit your answer should be a single number)

SOLUTION (a)

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y}(1 - e^{-x^2})(1 - e^{-y^2}) = 4xye^{-x^2-y^2}, \; x > 0, y > 0.$$

Then

$$P(X > 2Y) = \int_0^\infty \int_{2y}^\infty 4xye^{-x^2-y^2}dxdy = \int_0^\infty 4ye^{-y^2}\Big(\frac{1}{2}\int_{2y}^\infty e^{-x^2}d(x^2)\Big)dy$$
$$= \int_0^\infty 2ye^{-y^2}\int_{4y^2}^\infty e^{-t}dt = \int_0^\infty 2ye^{-5y^2}dy = \int_0^\infty e^{-5t}dt = \frac{1}{5}.$$

(b) Since $f_{XY}(x, y)$ is symmetric in x, y, we obtain

$$P(X > 2Y) + P(Y > 2X) = 2P(X > 2Y) = \frac{2}{5}.$$

Problem 5. (Markov chains)

Three players play a game in which they take turns and draw cards from an ordinary deck of 52 cards, successively, at random, and with replacement. Player I draws cards until an ace is drawn.

Then Player II draws cards until a diamond is drawn. Next, Player III draws cards until a face card is drawn. At that point, the deck is returned to Player I and the game continues. Determine the long-term proportion of cards drawn by each player out of the total number of cards drawn by the three players.

(A deck of cards consists of 4 suits: diamonds, hearts, clubs, and spades, 13 cards per suit, there are 3 face cards in each of the suits, so there are 4 aces, 13 diamond cards, and 12 face cards in the deck).

SOLUTION: The chain has 3 states, corresponding to the number of the player that uses the deck. The transitions are given by the following matrix:

$$\begin{pmatrix} 48/52 & 4/52 & 0 \\ 0 & 39/52 & 13/52 \\ 12/52 & 0 & 40/52 \end{pmatrix}$$

Our task is to solve the system $\pi P = \pi, \pi_1 + \pi_2 + \pi_3 = 1$.

The first equation of this system is $\pi_1 = \pi_2 \frac{48}{52} + \pi_3 \frac{12}{52}$, or $\pi_3 = \frac{1}{3}\pi_1$.

The second equation is $\pi_2 = \pi_1 \frac{4}{52} + \pi_2 \frac{39}{52}$, or $\pi_2 = \frac{4}{13}\pi_1$.

Substituting pi_2 and π_3 into $\pi_1 + \pi_2 + \pi_3 = 1$, we obtain $\pi_1 = 39/64$ and $\pi_2 = 12/64, \pi_3 = 13/64$, which are the required long-term proportions.

Problem 6. Let X be a random (uniform) number $x, 0 < x < 1$ and let $Y = X^2$. Are X and Y positively or negatively correlated, or uncorrelated? Please justify your answer by computing the correlation coefficient (no credit for just the answer or verbal argument).

SOLUTION: We have $f_X(x) = 1/x, 0 < x < 1$, and $EX = 1/2, EY = EX^2 = 1/3$,

$$\sigma_X^2 = EX^2 - (EX)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\sigma_Y^2 = EX^4 - (EX)^4 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45},$$

$$\mathrm{Cov}(X, Y) = EXY - (EX)(EY) = EX^3 - \frac{1}{6} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

Thus, X and Y are positively correlated, and

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{1/12}{1/2\sqrt{3} \cdot 2/3\sqrt{5}} = \frac{\sqrt{15}}{4}.$$