ENEE324. Final exam, 5/13/2026. Instructor: A. Barg

- The paper contains 6 problems. Each problem is 10 points. Max score=60 points
- Your answers should be justified. NO CREDIT for an answer with no justification.
- DO NOT copy the problem statement into your paper
- **Manage your time:** 20min/problem
- SIGN YOUR NAME!

Problem 1. A shipment contains 20 batteries, of which 5 are defective. An inspector randomly selects 6 batteries from the shipment without replacement.

Let X denote the number of defective batteries selected.

(1) Identify the distribution of X and find a general expression for $P(X = k)$. Identify the range of X. Note: $\binom{20}{6} = 38760$

(2) Compute the probability that exactly 2 defective batteries are selected. Simplify the result to obtain a numerical value.

(3) Compute the probability that at least three defective batteries are selected. Simplify the result to obtain a numerical value.

(4) Find the expected value and variance of X.

Solution: (1) Let $n = 6, w = 5, b = 15$, then the distribution of X is hypergeometric (w, b, n), and

$$P(X = k) = \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}, \quad k = 0, 1, 2, 3, 4, 5.$$

(2)

$$P(X = 2) = \frac{\binom{5}{2}\binom{15}{4}}{\binom{20}{6}} = \frac{10 \cdot 1365}{38760} = \frac{13650}{38760} \approx 0.352.$$

(3)

$$P(X \ge 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2) = 1 - \frac{\binom{15}{6}}{38760} - \frac{\binom{5}{1}\binom{15}{5}}{38760} - 0.352$$
$$= 1 - \frac{5005 + 5 \cdot 3003}{38760} - 0.352 \approx 0.131$$

(4) Let $N = w + b$. For $X \sim \text{HGeom}(w, b, n)$ let $I_i = \text{Ind}(i\text{th battery defective})$. Then $EI_i = \frac{w}{N}$ and $EX = n\frac{w}{N} = 1.5$. Next,

$$EX^2 = \sum_{k=1}^{5} k^2 P(X = k) \approx 3.079,$$

so $\text{Var}(X) \approx 3.079 - 2.25 = 0.829$.

Or, you can directly use the formula we found in Lec.22, getting the same answer:

$$\text{Var}(X) = \frac{N-n}{N-1} n \frac{w}{N}\frac{b}{N} = \frac{14}{19} 6 \frac{5}{20}\frac{15}{20} = \frac{63}{76} \approx 0.829$$

Problem 2. Let $X \sim \mathrm{FS}(p)$ and $Y \sim \mathrm{FS}(p)$ be independent **First Success** random variables with

$$P(X = k) = P(Y = k) = pq^{k-1}, \qquad k = 1, 2, \dots,$$

where $q = 1 - p$. Define

$$M = \max(X, Y), \qquad D = |X - Y|.$$

(1) Find $P(M \leq k)$ for $k = 1, 2, \dots$ and deduce the probability mass function $P(M = k)$ of M. Hint: Use independence and $\{M \leq k\} = \{X \leq k\} \cap \} Y \leq k\}$.
(2) Compute $P(D = 0)$ and $P(D = 2)$. Hint: $\{D = 2\} = \{X = Y + 2\} \cup \{Y = X + 2\}$
(3) Compute

$$P(X < Y).$$

Hint: $P(X < Y) + P(Y < X) + P(X = Y) = 1$
(4) Are the events $\{X < Y\}$ and $\{D = 0\}$ independent? Justify your answer.

Solution:

(1) For $k \geq 1$, using independence of X and Y,

$$P(M \leq k) = P(X \leq k,\ Y \leq k) = P(X \leq k)^2.$$

Now $P(X \leq k) = \sum_{i=1}^{k} pq^{k-1} = 1 - q^k$, hence $P(M \leq k) = (1 - q^k)^2$. Finally,

$$P(M = k) = P(M \leq k) - P(M \leq k - 1) = (1 - q^k)^2 - (1 - q^{k-1})^2.$$

(2) We have

$$P(D = 0) = P(X = Y).$$

Thus

$$P(D = 0) = \sum_{k=1}^{\infty} p^2 q^{2k-2} = \frac{p^2}{1 - q^2} = \frac{p}{2 - p}.$$

Next,

$$P(D = 2) = P(X = Y + 2) + P(Y = X + 2).$$

Hence

$$P(D = 2) = 2 \sum_{k=1}^{\infty} p^2 q^{2k} = \frac{2p^2 q^2}{1 - q^2} = \frac{2pq^2}{2 - p}.$$

(3) By symmetry,

$$P(X < Y) = P(Y < X).$$

Also,

$$P(X < Y) + P(Y < X) + P(X = Y) = 1.$$

Therefore,

$$2P(X < Y) + \frac{p}{2 - p} = 1,$$

so

$$P(X < Y) = \frac{1}{2}\left(1 - \frac{p}{2 - p}\right) = \frac{1 - p}{2 - p}.$$

(4) Since $\{D = 0\} = \{X = Y\}$, the events $\{X < Y\}$ and $\{D = 0\}$ are disjoint. Hence

$$P(\{X < Y\} \cap \{D = 0\}) = 0.$$

But

$$P(X < Y) > 0, \qquad P(D = 0) > 0.$$

Therefore the events are not independent.

Problem 3. Customers arrive at a service desk according to a Poisson random variable with mean 6 per hour. Let X denote the number of customers arriving during one hour.

(1) Given that at least one customer arrived during one hour, compute
$$P(X = 1 \mid X \geq 1).$$

(2) Compute the conditional expectation
$$\mathbb{E}[X \mid X \geq 1].$$

(3) Let
$$Y = \begin{cases} 1, & X \geq 5, \\ 0, & X < 5. \end{cases} \qquad \text{Compute } \mathbb{E}[Y].$$

(4) Suppose that each arriving customer independently makes a purchase with probability $1/3$. Let Z be the number of customers who make a purchase. Compute
$$P(Z = 0).$$

Solution:

Since $X \sim \text{Poisson}(6)$,
$$P(X = k) = e^{-6}\frac{6^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

(1) Using conditional probability,
$$P(X = 1 \mid X \geq 1) = \frac{P(X = 1)}{P(X \geq 1)}.$$

Since
$$P(X = 1) = 6e^{-6} \approx 0.0149, \qquad P(X \geq 1) = 1 - e^{-6} \approx 0.9975,$$
we obtain
$$P(X = 1 \mid X \geq 1) = \frac{6e^{-6}}{1 - e^{-6}} \approx 0.0149$$

(2)
$$\mathbb{E}[X \mid X \geq 1] = \sum_{k \geq 1} kP(X = k | X \geq 1) = \frac{\mathbb{E}[X\mathbf{1}_{\{X \geq 1\}}]}{P(X \geq 1)} = \frac{\sum_{k \geq 1} kP(X = k)}{P(X \geq 1)}.$$
$$= \frac{E(X)}{1 - e^{-6}} = \frac{6}{1 - e^{-6}} \approx 6.0149.$$

(3) Since $Y = \mathbf{1}_{\{X \geq 5\}}$,
$$\mathbb{E}[Y] = P(X \geq 5).$$

Therefore,
$$\mathbb{E}[Y] = 1 - P(X \leq 4) = 1 - e^{-6}\sum_{k=0}^{4}\frac{6^k}{k!} = 0.7149.$$

(4) The purchases "arrive" at Poisson rate $6 \cdot \frac{1}{3} = 2$, so
$$Z \sim \text{Poisson}\left(6 \cdot \frac{1}{3}\right) = \text{Poisson}(2).$$

Hence
$$P(Z = 0) = e^{-2} \approx 0.135.$$

Problem 4. A random point (X, Y) is chosen uniformly from the triangle

$$T = \{(x, y) : 0 < y < x < 1\}.$$

(1) Draw a sketch of the triangle in Cartesian coordinates and find the joint density function $f_{X,Y}(x, y)$.
(2) Find the marginal density functions of X and Y.
(3) Compute

$$\mathbb{E}[X], \qquad \mathbb{E}[Y], \qquad \mathbb{E}[XY].$$

(4) Compute

$$\mathrm{Cov}(X, Y).$$

(5) Are X and Y independent? Justify your answer.

Solution: The triangle $T = \{(x, y) : 0 < y < x < 1\}$ has area 1/2. Since (X, Y) is uniformly distributed on T,

$$f_{X,Y}(x, y) = \begin{cases} 2, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(1) The joint density is therefore

$$f_{X,Y}(x, y) = 2 \qquad (0 < y < x < 1).$$

(2) Marginal densities:
For $0 < x < 1$,

$$f_X(x) = \int_0^x 2\,dy = 2x.$$

For $0 < y < 1$,

$$f_Y(y) = \int_y^1 2\,dx = 2(1 - y).$$

Thus

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 2(1 - y), & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(3) Expectations:

$$\mathbb{E}[X] = \int_0^1 x(2x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}.$$

Similarly,

$$\mathbb{E}[Y] = \int_0^1 2y(1 - y)\,dy = \frac{1}{3}.$$

Also,

$$\mathbb{E}[XY] = \int_0^1 \int_0^x 2xy\,dy\,dx.$$

Compute the inner integral:

$$\int_0^x 2xy\,dy = x^3.$$

Hence

$$\mathbb{E}[XY] = \int_0^1 x^3\,dx = \frac{1}{4}.$$

(4) Covariance: $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. Therefore

$$\mathrm{Cov}(X,Y) = \frac{1}{4} - \frac{2}{3}\cdot\frac{1}{3} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}.$$

(5) X and Y are not independent because (1) either $\mathrm{Cov}(X,Y) \neq 0$ or because

$$f_{X,Y}(x,y) \neq f_X(x)f_Y(y).$$

Indeed,

$$f_X(x)f_Y(y) = 4x(1-y),$$

which is not equal to 2 on the triangle.

Problem 5. Let (X,Y) have joint density

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)}, & 0 < y < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$U = X - Y, \qquad V = Y.$$

(1) Find the inverse transformation expressing (X,Y) in terms of (U,V).
(2) Compute the Jacobian determinant.
(3) Determine the range of the pair of RVs (U,V).
(4) Find the joint density function $f_{U,V}(u,v)$.
(5) Are U and V independent? Justify your answer.

Solution: We have the transformation

$$U = X - Y, \qquad V = Y.$$

(1) Inverse transformation:

$$Y = V, \qquad X = U + V.$$

(2) Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1.$$

(3) Support: since $0 < y < x < \infty$,

$$0 < V < U + V \implies V > 0,\ U > 0.$$

Thus the support is

$$u > 0, \quad v > 0.$$

(4) Joint density:

$$f_{U,V}(u,v) = f_{X,Y}(u+v,v)\cdot|J| = 2e^{-((u+v)+v)} = 2e^{-u-2v}, \quad u,v > 0.$$

(5) Independence:

$$f_{U,V}(u,v) = 2e^{-u}e^{-2v} = (e^{-u})(2e^{-2v}).$$

This factorizes into a product of two **valid pdf functions**, namely Exp(1) and Exp(2), hence U and V are independent. (It is not enough to say that this is a product of some two random factors).

Problem 6. Let X be a random variable with moment generating function

$$M_X(t) = \left(\frac{2}{2-t}\right)^3, \qquad t < 2.$$

(1) Compute $\mathbb{E}[X], \mathrm{Var}(X)$.
(2) Find the moment generating function of

$$Y = 2X - 1.$$

(3) Let X_1, X_2 be independent random variables having the same distribution as X. Find the moment generating function of

$$S = X_1 + X_2.$$

Solution:

(1) Compute moments via derivatives.
First,

$$M_X'(t) = 3\left(\frac{2}{2-t}\right)^2 \cdot \frac{2}{(2-t)^2} = \frac{24}{(2-t)^4}.$$

Thus,

$$\mathbb{E}[X] = M_X'(0) = \frac{24}{16} = \frac{3}{2}.$$

[To compute $M_X'(0)$, **first** compute $\frac{d}{dt}M_X(t)$, **then** substitute $t = 0$]
Next,

$$M_X''(t) = \frac{96}{(2-t)^5}, \qquad \Rightarrow \mathbb{E}[X^2] = M_X''(0) = \frac{96}{32} = 3.$$

Hence

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 3 - \frac{9}{4} = \frac{3}{4}.$$

(2) For $Y = 2X - 1$,

$$M_Y(t) = \mathbb{E}[e^{t(2X-1)}] = e^{-t}M_X(2t).$$

Thus,

$$M_Y(t) = e^{-t}\left(\frac{2}{2-2t}\right)^3 = e^{-t}\left(\frac{1}{1-t}\right)^3.$$

(3) For $S = X_1 + X_2$ with independence,

$$M_S(t) = M_X(t)^2 = \left(\frac{2}{2-t}\right)^6.$$