ENEE324: Engineering Probability

Final Examination **December 17, 2016**

Each problem is worth 10 points, Max 60pts Instructor: A. Barg

- **Print Your First and Last Name!**
- **Clearly identify your answers to the questions!** No credit for an answer with no justification.

Problem 1. Each of three players is given a coin. In the first round of the game they toss their coins once, producing three outcomes. If one of these outcomes is different from the two other outcomes, then the game ends, otherwise they go into the second round.

(a) Assume that the coins are fair. What is the probability that the game ends after one round?

(b) Now assume that each of the coins is biased and shows H with probability 1/4. Under this assumption, what is the probability that the game ends after one round?

Solution: Let A be the needed event. Then

$$\text{(a)} \quad P(A) = 1 - P\{(HHH) \cup (TTT)\} = 1 - P(HHH) - P(TTT) = 1 - 2 \cdot \frac{1}{8} = 3/4$$

$$\text{(b)} \quad P(A) = 1 - \frac{1}{4^3} - \Big(\frac{3}{4}\Big)^3 = \frac{9}{16}.$$

Problem 2. The joint PMF of two discrete RVs is

$$p_{XY}(i,j) = \binom{j}{i} e^{-2\lambda} \lambda^j / j!, \quad 0 \le j < \infty, 0 \le i \le j$$

and 0 for all other i, j. Here $\lambda > 0$ is a fixed number.

(a) Find the PMF $p_Y(j)$ of Y (give the answer for all j). Your answer should not include any sums.

(b) Find the PMF $p_X(i)$ of X (give the answer for all i). Your answer should not include any sums.

(c) Are the RVs X and $Y - X$ independent? Please justify your answer.

(Useful formulas for (a),(b),(c): $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, $\sum_{k=0}^{\infty} \lambda^k / k! = e^{\lambda}$.)

Solution:

$$p_Y(j) = \sum_{i=0}^{j} p_{XY}(i,j) = e^{-2\lambda} \frac{\lambda^j}{j!} \sum_{i=0}^{j} \binom{j}{i} = e^{-2\lambda} \frac{(2\lambda)^j}{j!}, \ 0 \le j < \infty.$$

$$p_X(i) = \sum_{j=i}^{\infty} \binom{j}{i} e^{-2\lambda} \lambda^j / j! = e^{-2\lambda} \frac{1}{i!} \sum_{j=i}^{\infty} \frac{\lambda^j}{(j-i)!} = e^{-2\lambda} \frac{\lambda^i}{i!} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} = e^{-\lambda} \frac{\lambda^i}{i!}, \ 0 \le i \le \infty.$$

$$P(X = i, Y - X = k) = P(X = i, Y = k + i) = \binom{k+i}{i} e^{-2\lambda} \frac{\lambda^{k+i}}{(k+i)!} = e^{-\lambda} \frac{\lambda^i}{i!} \cdot e^{-\lambda} \frac{\lambda^k}{k!}.$$

Thus X and $Y - X$ are independent Poisson random variables.

Problem 3. Two continuous RVs are distributed according to the joint PDF

$$f_{XY}(x,y) = ax, \quad 1 \le x \le y \le 2 \text{ and } 0 \text{ o/w}.$$

(a) Find a;

(b) Find the marginal CDF $F_Y(y)$ (give the answer for all y);

(c) Find $E(\frac{1}{X} | Y = \frac{3}{2})$. Your answer should be a number and should not include x or y.

Solution: (a) $a = \dfrac{1}{\int_1^2 \int_x^2 x dy dx} = \dfrac{3}{2}.$

(b) $f_Y(y) = \int_1^y \frac{3}{2}x dx = 3(y^2-1)/4, 1 \le y \le 2$ and 0 otherwise. So

$$F_Y(y) = \begin{cases} 0 & y \le 1 \\ \int_1^y \frac{3}{4}(t^2-1)dt = (\frac{t^3}{4} - \frac{3}{4}t)|_1^y = \frac{1}{4}(y^3 - 3y + 2) & 1 < y \le 2 \\ 1 & y > 2 \end{cases}$$

(c) $f_{X|Y}(x|\frac{3}{2}) = \frac{f_{XY}(x,3/2)}{f_Y(3/2)} = \frac{8x}{5}, 1 \le x \le 3/2$ and so

$$E\Big(\frac{1}{X}\Big|Y = \frac{3}{2}\Big) = \int_1^{3/2} \frac{1}{x} \cdot \frac{8x}{5} dx = \frac{4}{5}.$$

Problem 4. Consider the Markov chain with transitions given by the following matrix:

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find the probability that in step 2 the chain is in state 4 given that it starts in state 2, i.e., find $P(X_2 = 4|X_0 = 2)$. Your answer should be a number.

(b) Identify recurrent and transient states, recurrent classes. Does the chain have a steady-state (stationary) distribution? If not, explain why.

(c) Assume that the chain starts in state 2. Compute the expected time till the first moment that the chain enters one of the recurrent classes. Your answer should be a number.
(Hint: For question (c) consider all the recurrent states as a single state).

Solution: (a)

$$P(X_2 = 4|X_0 = 2) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot 1 = \frac{7}{18}.$$

(b) States 1,2 are transient and states 3 and 4 each form a recurrent class of their own. Since there is more than one recurrent class, the steady-state distribution does not exist (the long-term behavior depends on where we start).

(c) Form a new chain with three states, where states 1 and 2 are the same as in the original chain, and state $R = \{3, 4\}$. Then the transition matrix of this new chain is

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

where the rows are numbered by 1,2,R, and the columns are numbered 1,2,R. Let $t_i, i = 1, 2$ be the expected time of getting from i to R. We have $t_2 = 1 + \frac{1}{3}t_1, t_1 = 1 + \frac{1}{4}t_1 + \frac{1}{4}t_2$ which gives $t_1 = \frac{15}{8}, t_2 = \frac{13}{8}$. Answer: $\frac{13}{8}$.

Problem 5. The number of hours between successive train arrivals is uniformly distributed on the interval $(0, 1)$. All the passengers on the platform board the arriving train, so once it leaves, the station is empty. Passengers arrive according to a Poisson process at rate 7 per hour. Suppose the train has just left the station. Let X be the random number of passengers that board the next train.

(a) Find EX

(Hints:
$X = N(T)$, where $N(T)$ is the RV that equals the number of arrivals in time T for the Poisson process.
To find EX, condition X on the random arrival time T of the next train.
Note that if $Y \sim$ Poisson(μ) then $EY = \text{Var}(Y) = \mu$ and $ET = \frac{1}{2}, \text{Var}(T) = \frac{1}{12}$.)

(b) Find Var(X).
Hint: for two RVs U and V we have $\text{Var}(U) = E(\text{Var}(U|V)) + \text{Var}(E(U|V))$.

Solution: We have $N(t) \sim \text{Poisson}(\lambda t)$, where $t \geq 0, \lambda = 7$. Thus, $E(N(T)|T) = \lambda T$, $\text{Var}\,(N(T)|T) = \lambda T$.

(a) $EX = E(N(T)) = E(E(N(T)|T)) = E(\lambda T) = 7/2$.

(b) $\text{Var}\,(X) = E(\text{Var}\,(X|T)) + \text{Var}\,(E(N(T)|T)) = E(\lambda T) + \text{Var}\,(\lambda T) = \lambda ET + \lambda^2 \text{Var}\,(T) = 7(\frac{1}{2} + 7\frac{1}{12}) = \frac{91}{12}$.

Problem 6. Let X be a continuous RV uniformly distributed on the interval (0,1). Let a, b be two positive numbers.

(a) Find the transform of the RV $Y = aX + b$.

(b) Using your answer in part (a), find the PDF $f_Y(y)$.

Solution: We have $M_X(t) = \int_0^1 e^{sx} dx = \frac{e^s - 1}{s}$, $s \neq 0$. Then

$$M_{aX+b}(s) = E(e^{s(aX+b)}) = e^{bs} E(e^{(as)X}) = e^{bs} \frac{e^{as} - 1}{as} = \frac{e^{(a+b)s} - e^{bs}}{((a+b) - b)s},$$

From this using uniqueness of transforms we obtain that $Y \sim \text{unif}(b, a+b)$, so

$$f_Y(y) = \frac{1}{a},\ b \leq y \leq a+b,\ \text{and } 0 \text{ o/w}.$$