ENEE324: Engineering Probability. Instructor: A. Barg

Final Examination

Problem 1. Let X be a uniform r.v. over the interval $(1,3]$.

(a) Determine the CDF and the PDF of the r.v. $Y = 1/X$.

(b) Using the Chebyshev inequality, estimate the probability $\mathbf{P}(|Y - \mathbf{E}[Y]| \ge 0.2)$.

Solution: (a) The range of Y is $[1/3, 1)$. So $f_Y(y) = 0$ for $y < 1/3$ and $y \ge 1$, and $F_Y(y) = 0$ for $y < 1/3$. Let $y \in [1/3, 1)$. Then

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\Big({}^1\!/_3 \le \frac{1}{X} \le y\Big) = \mathbf{P}(1/y \le X \le 3)$$
$$= \int_{1/y}^{3} \frac{dx}{2} = \frac{1}{2}\Big(3 - \frac{1}{y}\Big).$$

Answer:

$$F_Y(y) = \begin{cases} 0 & y < 1/3 \\ \frac{1}{2}\Big(3 - \frac{1}{y}\Big) & 1/3 \le y < 1 \\ 1 & y > 1. \end{cases} \qquad f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{1}{2y^2}, & 1/3 \le y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Compute $\mathbf{E}[Y] = \int_{1/3}^{1} \frac{dy}{2y} = {}^1\!/_2 \ln 3 \approx 0.55$, $\mathbf{E}[Y^2] = {}^1\!/_2 \int_{1/3}^{1} dy = \frac{1}{3}$, and $\sigma^2 = \frac{1}{3} - (0.55)^2 \approx 0.33 - 0.3025 = 0.0308$.

Then by Chebyshev,

$$\mathbf{P}(|Y - \mathbf{E}[Y]| \ge 0.2) \le \frac{\sigma^2}{0.04} = 0.77.$$

[Note that the true value is $\int_{0.35}^{0.55} f_Y(y)dy \approx 0.519$].

Answer: ≈ 0.77.

Problem 2. Trees of a particular rare species are located in a forest with an average density of one tree per $10000m^2$. Assume that the appearance of trees is described by a Poisson process.

(a) What is the probability that a $50000m^2$ plot will contain no fewer than three trees?

(b) Given a spot in the forest, find the probability that at least one tree is within 100m of this spot.

(c) Find the PDF of the random distance D between a tree of this kind and its nearest neighbor of the same kind.

Solution:

(a) This probability equals

$$1 - \mathbf{P}(\text{two or fewer arrivals}) = 1 - \sum_{s=0}^{2} \frac{(5\lambda)^s e^{-5\lambda}}{s!},$$

where $\lambda = 1$ arrival per unit area (the unit is $10000m^2$). Performing the calculation, we find the probability $1 - e^{-5}(1 + 5 + 25/2) \approx 0.875$.

(b) The area of the circle is $\pi(100m)^2 = \pi$ units. Then the required probability is

$$1 - \mathbf{P}(\text{no trees in a 100m circle}) = 1 - e^{-\lambda\pi} = 1 - e^{-\pi} \approx 0.9567.$$

(c) The distribution of the interarrival time is exponential. Let D be the r.v. such that the distance to the nearest neighbor equals $100D$ meters. Then $F_D(d) = 0, d \leq 0$, and

$$F_D(d) = \mathbf{P}(D \leq d) = 1 - e^{-\lambda \frac{\pi}{4} d^2} \quad (d > 0)$$

(the CDF of an exponential r.v.). For the PDF we then obtain

$$f_D(d) = 2\lambda d \frac{\pi}{4} e^{-\frac{\pi}{4} d^2} = {}^1\!/\!_2 \lambda \pi d e^{-\frac{\pi}{4} d^2} \quad (d \geq 0).$$

Answers: (a) 0.875, (b) 0.9567, (c), $f_D(d) = {}^1\!/\!_2 \lambda \pi d e^{-\frac{\pi}{4} d^2}$ for $d \geq 0$, zero otherwise.

Problem 3. An event A has probability $\mathbf{P}(A) = 0.7$. Suppose that the experiment is repeated 3 times and the number K of ocurrences of A is recorded. Next, an integer X is picked at random from the set $\{0, ..., K\}$.

(a) Find the PMF of K.

(b) Find the joint PMF $p_{X,K}(x, k)$.

(c) Find the marginal PMF $p_X(x)$.

Solution:

(a) $K \sim \text{Binom}(3, 0.7)$. Then

$$p_K(k) = \begin{cases} (0.3)^3 = 0.027 & k = 0 \\ 3 \cdot 0.7 \cdot (0.3)^2 = 0.189 & k = 1 \\ 3 \cdot (0.7)^2 0.3 = 0.441 & k = 2 \\ (0.7)^3 = 0.343 & k = 3 \end{cases}$$

(b)

$$p_{X,K}(x, k) = p_{X|K}(x|k) p_K(k) = \frac{1}{k+1} p_K(k), x = 0, \dots, k, k = 0, 1, 2, 3.$$

(c)

$$p_X(x) = \begin{cases} p_K(0) + \frac{1}{2} p_K(1) + \frac{1}{3} p_K(2) + \frac{1}{4} p_K(3) = 0.3543 & x = 0 \\ \frac{1}{2} p_K(1) + \frac{1}{3} p_K(2) + \frac{1}{4} p_K(3) = 0.3543 - p_K(1) = 0.3273 & x = 1 \\ 0.3273 - \frac{1}{2} p_K(1) = 0.2328 & x = 2 \\ 0.2378 - \frac{1}{3} p_K(2) = 0.08575 & x = 3. \end{cases}$$

Problem 4. Let $X \sim \text{Unif}[1, 2]$. Let Y be an r.v. given by its conditional PDF

$$\begin{cases} f_{Y|X}(y|x) = xe^{-xy}, & y \geq 0 \\ 0, & y < 0 \end{cases}.$$

(a) Compute $\mathbf{P}(Y > X)$.

(b) Find $\mathbf{P}(Y > X | X \geq 1.5)$.

Solution: (a) The joint PDF of X and Y is given by $f_{X,Y}(x, y) = xe^{-xy}$ for $1 \leq x \leq 2, y \geq 0$ and 0 otherwise. Then

$$\mathbf{P}(Y > X) = \int_{x=1}^{2} \int_{y=x}^{\infty} xe^{-xy} dy dx = \int_1^2 e^{-x^2} dx = \frac{1}{\sqrt{2}} \int_{\sqrt{2}}^{2\sqrt{2}} e^{-\frac{y^2}{2}} dy$$

$$= \sqrt{\pi}(\Phi(2\sqrt{2}) - \Phi(\sqrt{2})) \approx 0.1353.$$

(b) We have

$$\mathbf{P}(Y > X | X \geq 1.5) = \frac{\mathbf{P}(Y > X \text{ and } X \geq 1.5)}{\mathbf{P}(X \geq 1.5)}$$

$$= \frac{1}{1/2} \int_{x=1.5}^{2} \int_{y=x}^{\infty} xe^{-xy} = 2 \int_{x=1.5}^{2} e^{-x^2} dx = \frac{2}{\sqrt{2}} \int_{1.5\sqrt{2}}^{2\sqrt{2}} e^{-\frac{y^2}{2}} dy$$

$$= 2\sqrt{\pi}(\Phi(2\sqrt{2}) - \Phi(1.5\sqrt{2})) \approx 0.0518.$$

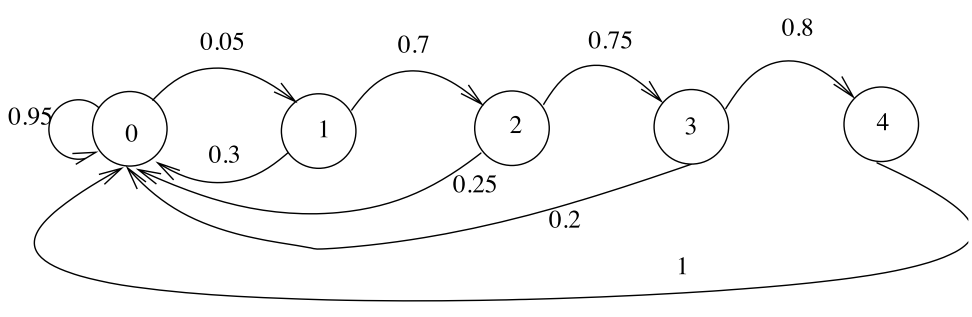
Problem 5. A DSL connection is normally functional, but occasionally gets broken with probability 1/20. If it is broken it can be restored within the next second with probability 0.3 or deteriorate further with probability 0.7. If the latter happens, then within the next second it can be restored with probability 0.25 or deteriorate even further with probability 0.75. If that happens, then within the next second it can recover completely with probability 0.2 or deteriorate even further with probability 0.8, from which state it is always restored within the next one-second slot.

(a) The behavior of the connection can be adequately described by (choose one)

(1) a sequence of binomial trials;
(2) a Markov chain;
(3) a Poisson process.

(b) If the answer to (a) is (1) or (3), stop here. If it is (2), compute the matrix of transition probabilities, the steady state distribution, and the expected time to see the connection restored, assuming that the system is initially in the second bad state, counting from the good state.

Solution: It is a Markov chain with states $0, 1, 2, 3, 4$ corresponding to a working connection, and the bad states of decreasing quality, respectively.



The transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} 0.95 & 0.05 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 & 0 \\ 0.25 & 0 & 0 & 0.75 & 0 \\ 0.2 & 0 & 0 & 0 & 0.8 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The chain has one recurrent class which is not periodic. Hence the stationary distribution exists and can be computed from the equations

$$\begin{gathered}\pi_0+\pi_1+\pi_2+\pi_3+\pi_4=1\\ 0.05\pi_0=\pi_1\\ 0.7\pi_1=\pi_2\\ 0.75\pi_2=\pi_3\\ 0.8\pi_3=\pi_4.\end{gathered}$$

From the last 4 equations,

$$\pi_4=0.8\pi_3=0.8\cdot 0.75\pi_2=0.8\cdot 0.75\cdot 0.7\pi_1=0.8\cdot 0.75\cdot 0.7\cdot 0.05\pi_0.$$

Using this in the normalization equation, we get

$$\pi_4\Big(1+\frac{1}{0.8}+\frac{1}{0.8\cdot 0.75}+\frac{1}{0.8\cdot 0.75\cdot 0.7}+\frac{1}{0.8\cdot 0.75\cdot 0.7\cdot 0.05}\Big)\approx 53.92\pi_4=1.$$

Then (upon rounding) $\pi_4=0.019, \pi_3=0.024, \pi_2=0.032, \pi_1=0.046, \pi_0=0.877$.

For the expected time to connection restoration, let t_i be the mean time to first get to 0 from state $i=2,3,4$ (note that state 1 is of no interest for this question because to get there from 2, we must first pass 0, but then we stop). Then

$$\begin{aligned}t_2&=1+0.75t_3\\ t_3&=1+0.8t_4\\ t_4&=1,\end{aligned}$$

from where we get $t_2=2.35$.

Answers: (a) a Markov chain. (b) The matrix **P** is as given above, the steady state distribution is $\pi_4=0.019, \pi_3=0.024, \pi_2=0.032, \pi_1=0.046\pi_0=0.877$, the expected wait time is 2.35sec.