ENEE324-03. Problem set 4

Date due: October 29, 2014

1. Let X be an r.v. with $f_X(x) = ae^{-0.02x}$ for $x \geq 0$ and 0 otherwise.

(a) Find a.

(b) Let $Y = 2X + 12$. Find $\mathbf{E}[Y]$.

(c) Find $\mathbf{P}(Y < 50)$.

(d) Find the CDF $F_Y(y)$.

2. Let $a < b$ be two real numbers. Let X be an r.v. with PDF

$$f_X(x) = \begin{cases} 0, & x \leq a \\ \frac{h}{b-a}x - \frac{ah}{b-a} & a < x \leq b \\ \frac{h}{a-b}x + \frac{(a-2b)h}{a-b} & b < x \leq 2b - a \\ 0 & x \geq 2b - a. \end{cases}$$

(a) Make a sketch of f_X. Express h as a function of a and b.

(b) Find $\mathbf{E}[X]$ and var(X) as a function of a and b (*i.e.*, the answer does not include h).

(c) Find the probability that on any given experiment the random variable will fall within one standard deviation from the mean (the correct answer does not depend on a and b, is just a number).

3. Let X be an r.v. with PMF $p_X(i) = \frac{1}{4}$ for $i = 1, 2, 3, 4$ and let Y be an r.v. given by the conditional PMF $p_{Y|X}(j|i) = \frac{1}{i}$ for $1 \leq j \leq i$ and $p_{Y|X}(j|i) = 0$ otherwise. (For instance, if $i = 2$, then $p_{Y|X}(1|2) = p_{Y|X}(2|2) = \frac{1}{2}$ and zero otherwise.)

(a) Find $p_{X,Y}(i, j)$.

(b) Find $p_Y(i)$.

(c) Find $p_{X|Y}(i|2)$.

(d) Find $\mathbf{E}[X|A]$, where A is the event $\{2 \leq Y \leq 3\}$.

4. The speed of a rolling ball is a random quantity $X(meters/sec)$ given by a random variable with CDF

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 0.5 + (0.1 - 0.005x)x, & 0 \leq x \leq 10 \\ 1, & x \geq 10. \end{cases}$$

(a) Make a sketch of $F_X(x)$.

(b) Find the PDF of X.

(c) Let the mass of the ball be 10 grams. Find $\mathbf{E}[K]$, where K is the kinetic energy of the ball measured in $g \cdot m^2/s^2$ (K equals half the mass times the square of the velocity).

(d) Find $f_K(x), F_K(x)$.

5. Let X be a Gaussian r.v., $X \sim \mathcal{N}(12, 10)$.

(a) Express $\mathbf{P}(X \leq 11)$ using the standard normal CDF Φ. Express the answer via $\text{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$.

(b) Using the standard normal table, find the numerical value of the answer in (a). (The table is in the textbook, p.155, or on the web).

6. A communication system transmits $+1$ or -1 in each one-second interval, independently of previous transmissions. The probability of transmitting $+1$ is p. The channel adds to the transmission a Gaussian $\mathcal{N}(0, \sigma^2)$ noise which is independent of the value of the transmitted signal.

(a) Suppose that the receiver decides for $+1$ if the received signal $Y \geq a$ and for -1 if $Y < a$, for some threshold a, $-1 \leq a \leq 1$. Write out the probability of an incorrect decision.

(b) Let $p = 1/3, a = 0.1, \sigma^2 = 1/2$. Find a numerical answer for the question of part (a).

7. Resistors are manufactured with a design mean of 1000 Ω and a standard deviation such that 20% fall outside the range $\pm 100\ \Omega$, for a yield of 80% working resistors. One day the machine was misadjusted, such that the mean became $1050\ \Omega$. The standard deviation remained the same as before. What was the new yield? Assume that the resistors have a Gaussian distribution prior to sorting.