ENEE324-03 Spring 2019. Problem set 8

Date due May 2, 2019

1. A stick of unit length is broken into three pieces by choosing two independent random points on the segment $(0,1)$ which serve as breakpoints. Denote the obtained pieces (as well as their lengths) by a, b, c. What is the probability that the segments a, b, c can be assembled to form a proper triangle?

Note that the segments form a triangle if and only if $a+b \ge c, a+c \ge b$, and $c+b \ge a$, and the answer can be obtained by a geometric argument without any integration.

2. Let X be a discrete RV with PMF

$$p_X(k) = 2\Big(\frac{1}{3}\Big)^k, k = 1, 2, 3, \dots.$$

Find the MGF (moment generating function) $g_X(s)$ and use the expression for $g_X(s)$ that you obtained to find EX.

3. The MGF of an RV X is equal to

$$g_X(s) = \frac{1}{3}e^s + \frac{4}{15}e^{3s} + \frac{2}{15}e^{4s} + \frac{4}{15}e^{5s}.$$

Find the PDF of X.

4. The MGF of an RV X is given by $g_X(s) = \frac{1}{(1-s)^2}, s < 1$. Find the moments EX^n for all $n = 1, 2, 3, \dots$.

5. Consider RVs X and Y with a joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}x^3 e^{-xy-x} & \text{if } x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y positively correlated, negatively correlated, or uncorrelated?

Hint: (The following integral will be useful: for all $a > 0$, $\int_0^\infty x^n e^{-ax} dx = n!/a^{n+1}$.)

6. Consider RVs X and Y with joint PDF given by

$$f_{XY}(x,y) = \begin{cases} (\sin x)(\sin y) & \text{if } 0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the correlation coefficient $\rho(X,Y)$.

(Markov chains)

7. Three white and three black balls are distributed in two urns, with three balls per urn. The state of the system is the number of white balls in the first urn. At each step, we draw at random a ball from each of the two urns, and exchange their places (the ball that was in the first urn is put into the second and vice versa).

(a) Determine the transition matrix for this Markov chain.

(b) Assume that initially all white balls are in the first urn. Determine the probability that this is also the case after 6 steps.

8. A Markov chain on the state space $S = \{0, 1, 2\}$ has the transition matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & 0 & \frac{1}{6} \end{bmatrix}$$

Assume that $P(X_0) = 0 = P(X_0 = 1) = \frac{1}{4}$. Determine the expected state of the chain after 3 transitions, EX_3.