ENEE324: Engineering Probability

Midterm Examination 1 **March 9, 2016**

Each problem is worth 10 points Instructor: A. Barg

• Please be precise and rigorous in your statemens, and show all your calculations. Please write neatly and legibly. Be sure to **Print Your Name!**

• This is a closed book exam, but you are allowed up to two 8.5× 11 pages of notes. No calculators please! Good luck!

Problem 1. Travelers buying airline tickets on a particular flight purchase economy class (40%), economy plus (35%) or business (25%). Of those in economy, 30% purchase duty-free goods in the onboard store, of those in economy plus 60% purchase duty-free, and of those in business 50%. Given that the next passenger bought duty-free, what is the probability that he is traveling in economy?

Solution: (Bayes formula)

$$P(E|D) = \frac{P(D|E)P(E)}{P(D)} = \frac{P(D|E)P(E)}{P(D|E)P(E) + P(D|E+)P(E+) + P(D|B)P(B)}$$

$$= \frac{0.3 \cdot 0.4}{0.3 \cdot 0.4 + 0.6 \cdot 0.35 + 0.5 \cdot 0.25} = \frac{24}{91}$$

Problem 2. A coin with $\Pr(H) = p$ is tossed repeatedly and independently until the first H is observed. Compute the probability of the event E that the first head appears in an even-numbered toss. (Hint: H appears in the first toss or not, and if it doesn't, then E is the event that H appears in an odd-numbered toss counting from toss 2. The probability of the last event equals $1 - P(E)$, and then the needed quantity is computed from the total probability formula.)

Solution 1: Let A_1 be the event that H appears in the first toss. We have

$$P(E) = P(E|A_1)P(A_1) + P(E|A_1^c)P(A_1^c). \tag{1}$$

Of course, $P(E|A_1) = 0$, and $E|A_1^c$ is the event that H appears in an odd toss if we start counting from toss 2. We obtain $P(E|A_1^c) = P(E^c) = 1 - P(E)$, and then from (1)

$$P(E) = 0 \cdot p + (1 - P(E)) \cdot (1 - p)$$

which gives $P(E) = (1 - p)/(2 - p)$.

Solution 2: The probability that H appears in toss $2k, k \geq 1$ is $(1 - p)^{2k-1}p$, so

$$P(E) = \sum_{k=1}^{\infty} (1 - p)^{2k-1} p = \frac{p}{1 - p} \sum_{k=1}^{\infty} ((1 - p)^2)^k = \frac{p}{1 - p} \cdot \frac{(1 - p)^2}{1 - (1 - p)^2}$$

$$= \frac{1 - p}{2 - p}.$$

Problem 3. Toss a fair coin 4 times and consider the random variable X indicating number of heads. Calculate $P[X = x|X \text{ even}]$ for $x = 0; 1; 2; 3; 4$.

Solution: Let $A = \{X \text{ even}\}$. We have

$$P(A) = \frac{1}{16}\left(\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\right) = \frac{1}{16}(1 + 6 + 1) = 1/2.$$

Then

$$P[X = x|X \text{ even}] = \frac{P[X = x \text{ and } X \text{ even}]}{P[X \text{ even}]}$$
$$= \begin{cases} 0, & x = 1,3 \\ 2P[X = x], & x = 0,2,4 \end{cases} = \begin{cases} 0, & x = 1,3 \\ \frac{1}{8}, & x = 0,4 \\ \frac{3}{4}, & x = 2 \end{cases}.$$

Problem 4. In an intersection, a car speeds to make the light with probability 0.2 and the decision to "go for it" is taken by different drivers independently. You have observed 6 cars passing through the intersection. Let X be the RV that equals the number of cars that sped to make the green light. (a) What is $EX, \operatorname{Var}(X)$? (b) Find the probability that $P(|X - EX| > 1)$ (approximate answer to within 10% is acceptable).

Solution: (a) X binomial with $n = 6, p = 0.2$, so $EX = 1.2, \operatorname{Var}(X) = 0.96$.

(b)

$$P(|X - EX| > 1) = 1 - P(|X - EX| \le 1) = 1 - P(X \in \{1, 2\}) = 1 - 6 \cdot 0.2 \cdot 0.8^5 - \binom{6}{2}(0.2)^2(0.8)^4$$
$$= 1 - 1.2 \cdot 0.32768 - 15 \cdot 0.04 \cdot 0.4096 \approx 1 - 0.38 - 0.25 = 0.37$$

(the exact value is 0.361024).

Problem 5. I have 10 different apps on my phone. Someone hacked into it, and was able to make an app start every 6 minutes, but not to control which app starts, so every 6 minutes a random one out of the 10 launches itself. Assume that, unless I start using the app just launched, it's immediately terminated, so they appear only for a very brief period of time.

(a) I decided to make use of this situation, and whenever I need an app I just wait for it to be launched. Suppose that I need app ♯1. Let X be the number of apps that will be started before ♯1 appears. What is the PMF of X, and what is its expected value and variance? Note that EX and $\operatorname{Var}(X)$ should be expressed as numbers.

(b) I took the phone out of my pocket at a random, uniformly distributed time during the day. What's the expected wait till the appearance of app ♯1?

Solution: (a) X is (almost) geometrically distributed: $p_X(i) = \left(\frac{9}{10}\right)^i \frac{1}{10}, i = 0, 1, \ldots$. The "almost" comes from the fact that geometric is number of trials till first success including the trial that constitutes the success, while X is one less than that number. Denoting this geometric by Y, we note that $X = Y - 1$, so

$$EX = EY - 1 = \frac{1}{p} - 1 = 9; \quad \operatorname{Var}(X) = \operatorname{Var}(Y) = \frac{1-p}{p^2} = \frac{9/10}{1/100} = 90.$$

(b) It will take $EU = 3$ minutes on average for the first app to appear, where U is an RV uniformly distributed on the segment $[0, 6]$. After that, according to part (a), app ♯1 will be the 10th on average, so the expected wait is $3 + 6 * 9 = 57$ min.