ENEE324: Engineering Probability

Midterm Examination 1 **March 14, 2019**

5 Problems, Each problem is worth 10 points

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• Please be precise and rigorous in your statemens, and show all your calculations. Please write neatly and legibly. Be sure to **Print Your Name!**

• The problem statements are printed on both sides of this sheet.

• This is a closed book exam, but you are allowed up to two 8.5× 11 pages of notes. No calculators please! Good luck!

Problem 1. (Please give your answers in the simplest possible form)

There are 3 sports teams, each consisting of n athletes. From this group of $3n$ people, a group of 4 athletes is chosen.

(a) How many choices are possible?

(b) How many choices are there if exactly three of the chosen athletes are on the same team?

(c) How many choices are there in which one pair out of the four chosen athletes share the same team, and the other pair also share the same team, different from the first one?

(d) How many choices if exactly three of the four chosen athletes are in different teams?

SOLUTION:

(a) $\binom{3n}{4}$

(b) $3\binom{n}{3} \cdot 2n = 6n\binom{n}{3}$ (choose the team with 3; choose 3 athletes in it; choose the team for the remaining athlete; choose the athlete in that team)

(c) $\binom{3}{2}\binom{n}{2}\binom{n}{2} = \frac{3}{4}n^2(n-1)^2$. (choose the two teams and the athletes in them)

(d) $3\binom{n}{2}n^2 = \frac{3}{2}n^2(n-1)$. (choose the team with 2 athletes, choose two athletes in it, choose one out of each in the two remaining teams).

Problem 2. We are given a random variable with CDF

$$F_X(x) = \begin{cases} 0 & x < -3 \\ k/8 & -3 \le x < 0 \\ k/4 & 0 \le x < 6 \\ 1 & x \ge 6. \end{cases}$$

(a) Is X continuous or discrete?

(b) Find k, if it is known that $p_X(0) - p_X(6) = \frac{1}{8}$.

(c) Find $EX, \mathrm{Var}(X)$.

(The answers for k, EX, and $\mathrm{Var}(X)$ each should be a number.)

SOLUTION: (a) The RV X takes only 3 values, so it is discrete. (b) We have $p_X(-3) = \frac{k}{8}, p_X(0) = F_X(0) - p_X(-3) = \frac{k}{8}, p_X(6) = \frac{k}{8} - \frac{1}{8}$,

$$1 = p_X(-3) + p_X(0) + p_X(6) = \frac{3k}{8} - \frac{1}{8},$$

and $k = 3$.

(c) From (b), the PMF of X is

k	-3	0	6
$p_X(k)$	3/8	3/8	1/4.

Thus, $EX = -9/8 + 12/8 = 3/8, EX^2 = 27/8 + 72/8 = 99/8, \text{Var}(X) = 99/8 - 9/64 = 783/64$.

Problem 3. A coin is tossed indefinitely. Assume that the probability that a toss yields H equals $p \in (0, 1)$.

(a) Let A_n be the event that in the first n tosses there was exactly one H. Let E_i be the event that the ith toss gave an H, where i is some fixed index in the set $\{1, 2, \dots, n\}$. Find $P(E_i|A_n)$.

(b) Now assume that the coin is tossed indefinitely, yielding a sequence of H and T. Find the probability that the first H is observed in an odd-numbered toss counting from the beginning of the experiment (please give the answer that does not include infinite sums).

SOLUTION:

(a) For any $i = 1, 2, \dots, n$

$$P(E_i|A_n) = \frac{P(E_i A_n)}{P(A_n)} = \frac{p(1-p)^{n-1}}{np(1-p)^{n-1}} = \frac{1}{n}.$$

(b) For any $k \ge 0$, the event $\{2k$ tails followed by H$\}$ has probability $(1-p)^{2k}p$. The required probability equals

$$\sum_{k=0}^{\infty}(1-p)^{2k}p = p\frac{1}{1-(1-p)^2} = \frac{1}{2-p}.$$

Problem 4. (In each question your answer should be a single number. It is possible to approximate the answers accurately without calculators.)

A company is flying 5 planes, and each of the five planes flies once on each given day. The probability that a flight is delayed is 0.2, independently of the other flights.

(a) On a given day we know that plane A was delayed, and do not have any other information about the remaining flights. What is the probability of this event?

(b) On a given day we know that one (unspecified) plane was delayed, and do not have any other information about the remaining flights. What is the probability of this event?

(c) Conditional on the event described in part (b), what is the probability that exactly two other (unspecified) planes were delayed on that day? In this question we are not concerned with the identity of the delayed planes.

SOLUTION: (a) 0.2 (b) Let $X \sim \text{Binom}(5, 0.2)$, then $P(X \ge 1) = 1 - P(X = 0) = 1 - (0.8)^5 \approx 0.67$.

(c)

$$P(X = 3|X \ge 1) = \frac{P(X = 3)}{P(X \ge 1)} \approx \frac{\binom{5}{3}0.2^3 0.8^2}{0.67} = \frac{10 \times 0.008 \times 0.64}{0.67} \approx 0.076.$$

Problem 5. Let X be a random variable uniformly distributed on the segment $(0,1)$.

(a) Determine the range of values of the RV $Y = -\ln(1-X)$.

(b) Find $E(Y)$.

Your answer in Part (b) should be a single number.

SOLUTION:

Part (a) The range of Y is from 0 to $+\infty$ $(-\ln(1-0)=0; -\ln(1-x) \xrightarrow{x\to 1} \infty)$

Part (b) FIRST SOLUTION: If $Y = g(X)$ and $f_X(x)$ is the PDF of X, then $EY = \int_{\infty}^{\infty} g(x) f_X(x)dx$. We have $f_X(x) = 1$ for $0 < x < 1$ and 0 o/w, and

$$\int_0^1 (-\ln(1-x))dx = \Big[(1-x)\ln(1-x) - (1-x)\Big]_0^1 = 1.$$

Part (b) SECOND SOLUTION: Compute

$F_Y(x) = P(Y \le x) = P(\ln(1-X) \ge -x) = P(1-X \ge e^{-x}) = P(X \le 1-e^{-x}) = F_X(1-e^{-x}) = 1-e^{-x}$, if $x \ge 0$, and 0 otherwise. Now $f_Y(x) = F_Y'(x) = e^{-x}, x \ge 0$ and

$$EY = \int_0^{\infty} xe^{-x}dx = -xe^{-x}|_0^{\infty} + \int_0^{\infty} e^{-x}dx = 1$$

(Or notice that $Y \sim \text{Exp}(\lambda)$, where $\lambda = 1$, and $EY = \frac{1}{\lambda} = 1$.)