ENEE324: Engineering Probability

Midterm Examination 2 **April 18, 2016**

Each problem is worth 10 points Instructor: A. Barg

• Please be precise and rigorous in your statements, and show all your calculations. Please write neatly and legibly. Be sure to **Print Your Name!**

• This is a closed book exam, but you are allowed up to two 8.5× 11 pages of notes. No calculators please! Good luck!

Problem 1. (a) Let X be a uniform RV over the interval $[1, 3]$. What are the CDF and PDF of the RV $Y = 1/X$?

(b) Let X be a uniform RV over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Find the PDF of the RV $Y = \tan X$. What is the expectation EY?

Solution: (a) The range of Y is $[1/3, 1]$; $f_Y(y) = 1/2(u(1/3)-u(1))$. Take y in the interval $[1/3, 1]$, then

$$F_Y(y) = \int_{\frac{1}{x}\le y} \frac{dx}{2} = \frac{1}{2}\int_{\frac{1}{y}}^{3} dx = \frac{1}{2}\Big(3-\frac{1}{y}\Big),\ \frac{1}{3} < y \le 1;\ 0 \text{ if } y \le 1/3; 1 \text{ if } y > 1.$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2y^2} \ \text{ if } \frac{1}{3} \le y \le 1;\ 0 \text{ o/w.}$$

(b) The range of Y is $[-\infty, \infty]$. Let $y \in \mathbb{R}$, then

$$F_Y(y) = \int_{x:\tan x\le y} \frac{dx}{\pi} = \int_{-\frac{\pi}{2}}^{\tan^{-1}(y)} \frac{dx}{\pi} = \frac{\tan^{-1}(y)}{\pi} + \frac{1}{2}, \quad -\infty < y < \infty$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{\pi(1+y^2)},\ -\infty < y < \infty.$$

The expectation $EY = \int\limits_{-\infty}^{\infty} \frac{ydy}{\pi(1+y^2)} = \frac{1}{2\pi}\ln(1+y^2)\Big|_{-\infty}^{\infty} = \infty - \infty$ is undefined (although this is not needed to answer this question, this PDF is known as the Cauchy distribution).

Problem 2. Given an RV $X \sim \mathcal{N}(\mu, \sigma^2)$, consider the RV Z that equals the magnitude of the deviation of X from its mean. Find the CDF of Z (express your answer using the standard normal CDF). What is the PDF of Z?

Solution: We have $Z = |X - \mu|$, $F_Z(z) = 0, z \le 0$, and

$$F_Z(z) = P(|X-\mu| \le z) = P(-z+\mu \le X \le z+\mu) = P\Big(-\frac{z}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{z}{\sigma}\Big)$$

$$= 2\Phi\Big(\frac{z}{\sigma}\Big) - 1, \quad 0 \le z \le \infty.$$

Thus

$$f_Z(z) = F_Z'(z) = \begin{cases} 0 & \text{if } z \le 0 \\ \frac{2}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}} & z > 0. \end{cases}$$

Problem 3. The joint PDF of the RVs X and Y is

$$f_{XY}(x,y) = \frac{e^{-y}}{y}, \quad 0 < x < y, 0 < y < \infty.$$

Find $E[X^3|Y=y]$.

Solution:

$$f_Y(y) = \int_0^y \frac{e^{-y}}{y} dx = e^{-y}, \quad y \geq 0; \qquad f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{y}, \quad 0 < x < y.$$

$$E[X^3|Y=y] = \int_0^y x^3 f_{X|Y}(x|y)dx = \int_0^y \frac{x^3}{y} dx = \frac{y^3}{4}.$$

Problem 4. In a town, there are two taxi companies. Suppose that the number of taxi cabs of the first company that passes a particular location in the street during some time interval is a Poisson random variable X with parameter λ_1. For the second company, the number of their cabs passing this location during the same time interval is a Poisson RV Y with parameter λ_2. Assume that X and Y are independent and let Z be the total number of cabs that drive past this location during the same time interval.

(a) Find the PMF of Z using only the PMF's of X and Y. What type of distribution is it? (Identify the distribution, e.g., binomial (n,p) or similar.)

(b) Now find the PMF of Z using only transforms. Justify the steps you made in obtaining your answer.

Solution: (a) Let $n \geq 0$ be integer.

$$p_Z(n) = \sum_{i=0}^{n} p_X(i) p_Y(n-i) = \sum_{i=0}^{n} e^{-(\lambda_1+\lambda_2)} \frac{\lambda_1^i \lambda_2^{n-i}}{i!(n-i)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \lambda_1^i \lambda_2^{n-i} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n.$$

Thus, Z is a Poisson RV with parameter $\lambda_1 + \lambda_2$.

(b) Since X and Y are independent, we have for all $s \in \mathbb{R}$

$$M_Z(s) = M_X(s) M_Y(s) = e^{\lambda_1(e^s-1)} e^{\lambda_2(e^s-1)} = e^{(\lambda_1+\lambda_2)(e^s-1)}.$$

Using the uniqueness of transforms, we again argue that $Z \sim \text{Poisson}(\lambda_1+\lambda_2)$.

Problem 5. Amtrak Train 1 arrives at Union Station in DC at a random time between 11:00 A.M. and 12:30 P.M. tomorrow. Train 2 arrives at the same station at a random time between 11:00 A.M. and the arrival time of train 1. Find the expected arrival time of train 2. Your answer should be a specific time value (such as 9:10 AM).

Solution: Take X to be the gap in minutes between 11am and the arrival of train 1, $X \sim \text{unif}[0, 90]$. For a given x, $Y \sim \text{unif}[0, x]$. We have

$$EY = E[E[Y|X]] = \int_0^{90} E[Y|X=x] f_X(x) dx = \int_0^{90} \frac{x}{2} \frac{dx}{90} = \frac{90^2}{4 \cdot 90} = 22.5.$$

Answer: 11:22:30 AM.

If you assumed that the interval is 25hrs 30min, then the above calculation would give 25.5hrs/4=6.375 hrs=6 h 22m 30 s, so you would obtain the answer 5:22:30PM. This is also fine.