ENEE324: Engineering Probability

Midterm Examination 2 **April 23, 2019**

4 Problems, Each problem is worth 10 points

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• Please be precise and rigorous in your statements, and show all your calculations. Please write neatly and legibly. Be sure to **Print Your Name!**

• The problem statements are printed on both sides of this sheet.

• This is a closed book exam, but you are allowed up to two 8.5× 11 pages of notes. No calculators please! Good luck!

Problem 1. In this problem the answers should be given by expressions for the PDFs that do not involve any sums or integrals.

(a) (4) Customer Bob enters the bank while another customer, Alice, is being served by a clerk. The service times of the customers, denoted C_A and C_B, are independent exponential random variables with the common parameter λ.

Once Alice leaves, the clerk attends to Bob. What is the PDF of the total time that Bob spends in the bank from the moment he enters the bank to the moment that he exits the bank upon completion of his service?

(b) (6) In this question Customers A and B enter the bank simultaneously, and there are two clerks available, so both A and B are served without waiting, each by their own clerk. The service times of Customers A and B are independent exponential random variables with a common parameter λ. Let T be the time that the customer *served first* (of these two customers) leaves the bank. Let S be the time that the customer *served last* leaves the bank. (Note that $T = \min(C_A, C_B), S = \max(C_A, C_B)$.)

What is the PDF of the random variable T?

What is the PDF of the random variable S?

SOLUTION:

(a) The time of the remaining service for Alice from the moment that Bob enters the bank till the end of her service is exponentially distributed with parameter λ (it does not matter that her service started before Bob entered the bank because exponential random variables are memoryless). Therefore, we set $X = C_A + C_B$, and the PDF of X is our answer. It is computed as follows:

$$f_X(x) = \int_0^x f_{C_A}(x-t) f_{C_B}(t)dt = \int_0^x (\lambda e^{-\lambda(x-t)})(\lambda e^{-\lambda t})dt = \lambda^2 x e^{-\lambda x},\ x > 0.$$

(b) Below we use the expression for the CDF of an exponential RV, $F(x) = 1 - e^{-\lambda x}$. Let $T = \min(C_A, C_B)$, then

$$P(T \le t) = 1 - P(T > t) = 1 - P(C_A > t, C_B > t) = 1 - P(C_A > t)P(C_B > t) = 1 - e^{-2\lambda t}, t > 0.$$

Thus, $T \sim \text{Exp}\,(2\lambda)$, $f_T(t) = \frac{d}{dt}[P(T \le t)] = 2\lambda e^{-2\lambda t}, t > 0.$

Similarly,

$$P(S \le t) = P(\max(C_A, C_B) \le t)) = P(C_A \le t, C_B \le t) = P(C_A \le t)P(C_B \le t) = (1 - e^{-\lambda t})^2, t > 0$$

$$f_S(t) = \frac{d}{dt}[P(S \le t)] = 2\lambda e^{-\lambda t}(1 - e^{-\lambda t}), t > 0.$$

Problem 2. Let X and Y be continuous random variables with joint probability density function

$$f_{XY}(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & \text{if } 0 < x < 1,\, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the conditional PDF $f_{X|Y}(x|y)$. Your answer should be an explicit expression that does not involve any sums or integrals.

(b) Find the conditional expectation $E(X^2|Y = 1/3)$. Your answer in Part (b) should be a single number.

SOLUTION: (a) Since $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$, we begin with computing $f_Y(y)$:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_0^1 \frac{3}{2}(x^2+y^2)dx = \frac{3}{2}y^2 + \frac{1}{2},\ 0 < y < 1$$

and zero elsewhere. Then

$$f_{X|Y}(x|y) = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}y^2+\frac{1}{2}} = \frac{3(x^2+y^2)}{3y^2+1},\ 0 < x < 1$$

and zero elsewhere.

(b) For $Y = 1/3$ we obtain $f_{X|Y}(x|\frac{1}{3}) = \frac{3x^2+1/3}{4/3} = \frac{3}{4}(3x^2+1/3)$. Then

$$E(X|Y = 1/3) = \int_0^1 x^2 f_{X|Y}(x|1/3)dx = \frac{3}{4}\int_0^1 (3x^4 + \frac{1}{3}x^2)dx = \frac{3}{4}\Big(\frac{3}{5} + \frac{1}{9}\Big) = \frac{8}{15}$$

Problem 3. The joint PDF of continuous random variables X, Y, Z has the form

$$f_{XYZ}(x,y,z) = \begin{cases} c(x+y+2z) & \text{if } 0 \le x,y,z \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the numerical value of c;

(b) Find $P(X < 1/3|Y < 1/2, Z < 1/4)$. Your answer should be a single number.

SOLUTION: (a)

$$\int_{x,y,z=0}^{1} (x+y+2z)dxdydz = \frac{1}{2} + \frac{1}{2} + 1 = 2,$$

so $c = 1/2$.

(b)

$$2P(Y < 1/2, Z < 1/4) = \int_0^1\int_0^{1/2}\int_0^{1/4}(x+y+2z)dxdydz = \int_{x,y,z} xdxdydz + \int_{x,y,z} ydxdydz + \int_{x,y,z} 2zdxdydz$$

$$= \frac{x^2}{2}\Big|_0^1 \cdot y|_0^{1/2} \cdot z|_0^{1/4} + x|_0^1 \cdot \frac{y^2}{2}\Big|_0^{1/2} \cdot z|_0^{1/4} + x|_0^1 \cdot y|_0^{1/2} \cdot z^2|_0^{1/4} = \frac{1}{8}.$$

$$2P(X < 1/3, Y < 1/2, Z < 1/4) = \int_0^{1/3}\int_0^{1/2}\int_0^{1/4}(x+y+2z)dxdydz = \frac{1}{96}$$

and thus

$$P(X < 1/3|Y < 1/2, Z < 1/4) = \frac{P(X < 1/3, Y < 1/2, Z < 1/4)}{P(Y < 1/2, Z < 1/4)} = \frac{1}{12}.$$

Problem 4. A fair 6-sided die with faces labelled from 1 to 6 is rolled 100 times. Let K be the random number of even outcomes among these 100 rolls.

(a) Give an explicit expression for the probability mass function $p_K(k)$ of the RV K.

(As an example, $p_X(i) = e^{-2}2^i/i!, i \geq 0$ is an explicit expression while $p_X(i) = P(X = i)$ is not. This PMF does not represent the answer to this question).

Use the expression $p_X(k)$ that you found to write an explicit expression for the probability $P(46 \leq K \leq 53)$. You do not have to compute the final numerical answer.

(b) Write an approximate expression for the probability $P(46 \leq K \leq 53)$ using the standard normal CDF Φ. Please perform all the computations except for the value of Φ itself (the answer of the form $\Phi(0.2)$ is acceptable, the answer of the form $\Phi((X-a)/b)$ is not).

SOLUTION: (a) The random variable K is binomially distributed with $P(K = 1) = \frac{1}{2}$, so $p_K(k) = \binom{100}{k}2^{-100}, k = 0, 1, \ldots, 100$, and

$$P(46 \leq K \leq 53) = \sum_{i=46}^{53} \binom{100}{k} 2^{-100}.$$

(b) We have $EK = 50, \text{Var}(K) = 25$, and thus the random variable $\frac{K-50}{5}$ is approximately standard Gaussian. Thus,

$$P(46 \leq K \leq 53) \approx \Phi(3/\sqrt{\text{Var}(K)}) - \Phi(-4/\sqrt{\text{Var}(K)}) = \Phi(0.6) - \Phi(-0.8).$$