ENEE620. Midterm examination, 11/16/2023. Instructor: A. Barg

- Please submit your work to ELMS Assignments as a single PDF file by Nov.16, 6:00pm EDT.
- Each problem is 10 points. Max score=50 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1. Let $X_n, n \ge 1$ be a sequence of i.i.d. RVs with $EX < \infty$, such that their sum $S_n = X_1 + \dots + X_n, n \ge 1$ satisfies $\frac{1}{n} S_n \overset{\text{a.s.}}{\to} C$, where $C$ is a constant. In this problem our goal is to show that $C = EX$.

(a) First prove that

$$\frac{X_n}{n} = \frac{S_n}{n} - \Big(\frac{n-1}{n}\Big)\frac{S_{n-1}}{n-1} \overset{\text{a.s.}}{\to} 0$$

Conclude that $P(|X_n| \ge n \text{ i.o.}) = 0$. Please give a rigorous argument.

(b) Then prove that for any fixed $i \ge 1$

$$\sum_{n=1}^{\infty} P(|X_i| \ge n) < \infty.$$

(c) Prove that, if $Z \ge 0$ is a random variable that takes nonnegative but not necessarily only integer values, then

$$\sum_{n=1}^{\infty} P(Z \ge n) \le EZ \le 1 + \sum_{n=1}^{\infty} P(Z \ge n).$$

Conclude therefore that $E|X| < \infty$ and deduce the needed claim about $C = EX$, with justification.

Solution:

(a) Taking the limit,

$$\lim_{n\to\infty} \Big(\frac{S_n}{n} - \Big(\frac{n-1}{n}\Big)\frac{S_{n-1}}{n-1}\Big) \overset{\text{a.s.}}{\to} C - 1 \cdot C = 0.$$

Thus we obtain that $\frac{X_n}{n} \overset{\text{a.s.}}{\to} 0$, which implies that $\frac{|X_n|}{n} \overset{\text{a.s.}}{\to} 0$. Thus, for some $N$ and all $n \ge N$, $\frac{|X_n|}{n} < 1$ a.s. In other words,

$$P\Big(\bigcup_{N\ge 1}\bigcap_{n\ge N}\Big\{\frac{|X_n|}{n} < 1\Big\}\Big) = 1.$$

Taking the complementary event, $1 - P(\limsup_n\{|X_n|/n > 1\}) = 1$, or $P(\limsup_n\{(|X_n|/n > 1\}) = 0$, i.e., $P(|X_n| \ge n \text{ i.o.}) = 0$, as required.

(b) The Borel-Cantelli lemma says that, for independent events $A_n$, if $\sum_n P(A_n) = \infty$, then $P(A_n \text{ i.o.}) = 1$. This is the same as the claim that $P(A_n \text{ i.o.}) < 1$ implies that $\sum_n P(A_n) < \infty$. Now take the events $E_n = \{|X_n| > n\}$, and note that they are independent. We conclude that

$$P(\limsup_n\{|X_n| > n\} = 0) < 1 \quad \Rightarrow \quad \sum_{n\ge 1} P(|X_n| > n) < \infty.$$

Since $X_n$ are i.i.d., we can replace $X_n$ by a generic $X$ on the previous line, or by any fixed $X_i$.

(c) Put $A_n = \{n-1 \le Z < n\}$ and note that

$$\sum_{n=1}^{\infty}(n-1)\mathbb{1}_{A_n} \le Z < \sum_{n=1}^{\infty} n\mathbb{1}_{A_n}$$

Taking expectations, we obtain on the left $\sum_{n=1}^{\infty}(n-1)P(A_n) = \sum_{n=1}^{\infty} P(Z \ge n)$, and on the right $\sum_{n=1}^{\infty} n\mathbb{1}_{A_n} = 1 + \sum_{n=1}^{\infty} P(Z \ge n)$.This proves the inequalities

$$\sum_{n=1}^{\infty} P(Z \ge n) \le EZ \le 1 + \sum_{n=1}^{\infty} P(Z \ge n).$$

Now Part (b) implies that $E|X| < \infty$, so the sequence $(X_n)$ satisfies the assumptions of the SLLN. In other words, $\frac{1}{n}S_n \overset{\text{a.s.}}{\to} EX$, and since the limit is unique, $C = EX$.

Problem 2.

There are $n$ white balls and $n$ black balls in a box. We repeatedly draw a random ball out of the box, without replacement. If the ball is white, we gain one unit of money, if it is black, our current capital does not change. Let $X_i$ be the amount of money we have after the $i$th draw.

(a) Show that the sequence $Y_i = \frac{2X_i - i}{2n-i}, 1 \le i \le 2n-1$ forms a martingale with respect to the filtration $\mathcal{F}$ given by $\mathcal{F}_i = \sigma(X_1, \dots, X_i), i \ge 1$.

(b) Find the expectation $EY_i$ for all $i$.

(c) Show that the sequence $Z_i = \frac{2n-i}{2n-i-1}Y_i^2 - \frac{1}{2n-i-1}, i = 1, 2, \dots, 2n-2$ forms a martingale with respect to $(\mathcal{F}_i)_i$.

Solution:

(a) Clearly the sequence $Y_i$ is integrable and adapted to the filtration $(\mathcal{F}_i)$.

Suppose that $X_i = k$, so there are $n-k$ white balls and $n-i+k$ black balls left in the box. We find

$$E(X_{i+1}|X_i = k) = \frac{(k+1)(n-k) + k(n-i+k)}{2n-i}.$$

Note that this expectation does not depend on the values of $X_{i-1}, X_{i-2}, \dots$ since the process $(X_i)_i$ has Markov property. Then assume that $X_i = k$ and find (after simplifications)

$$E(Y_{i+1}|X_i = k) = \frac{2E(X_{i+1}|X_i = k) - i - 1}{2n-i-1} = \frac{2k-i}{2n-i}.$$

Thus,

$$E(Y_{i+1}|\mathcal{F}_i) = \frac{2X_i - i}{2n-i} = Y_i.$$

(b) By the martingale property, $EY_i = EY_1 = \frac{2E(X_1)-1}{2n-1}$, and since $EX_1 = \frac{1}{2}$, $EY_i = 0$ for all $i$.

(c) As in part (a), the sequence $(Z_i)_i$ is integrable. We have

$$\begin{aligned} E(X_{i+1}^2|X_i = k) &= (k+1)^2 P(X_{i+1} = k+1|X_i = k) + k^2 P(X_{i+1} = k|X_i = k) \\ &= \frac{(k+1)^2(n-k)}{2n-i} + \frac{k^2(n-i+k)}{2n-i} \end{aligned}$$

Then

$$E[Y_{i+1}^2|X_i = k] = \frac{E[(2X_{i+1} - i - 1)^2|X_i = k]}{(2n-i-1)^2} = \frac{4E[X_{i+1}^2|X_i = k] - 4(i+1)E[X_{i+1}|X_i = k] + (i+1)^2}{(2n-i-1)^2}$$

We have earlier computed both expectations in the numerator, so it remains to substitute and simplify, and we obtain

$$E[Z_{i+1}|X_i = k] = \frac{(i-2k)^2 + i - 2n}{(i-2n)(i-2n+1)}$$

At the same time, if $X_i = k$, then

$$Z_i = \left[\frac{(2n-i)\left(\frac{2k-i}{2n-i}\right)^2}{2n-i-1} - \frac{1}{2n-i-1}\right] = \frac{(i-2k)^2 + i - 2n}{(i-2n)(i-2n+1)}.$$

Thus for every value of $X_i$ the expressions coincide, and therefore, $E[Z_{i+1}|X_i] = Z_i$ a.s.

Problem 3.

Consider a Markov chain with the state space $S = \{1, 2, \dots\}$ and transition probabilities given by $p_{12} = 1$

and

$$p_{ij} = \begin{cases} i^{-a} & \text{if } j = 1 \text{ and } i \ge 2 \\ 1 - i^{-a} & \text{if } j = i+1 \text{ and } i \ge 2 \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $a > 0$ is some number.

($a_1$) Take $a = 1$ and find the probability $f_{11}(n)$ of returning to state 1 in $n$ steps, $n \ge 1$. Is state 1 recurrent? If yes, compute the expected time of return. If state 1 is recurrent, is it positive or null recurrent?

($a_2$) Does this chain have a limiting distribution? In particular, in the long run, what is the proportion of time that the chain will spend in state 1?

(b) Keeping $a = 1$, flip the first two cases in the definition of $p_{ij}$ in Eq. (1) (i.e., take $p_{ij} = 1 - (1/i)$ for $j = 1$ and $i \ge 2$ and $p_{ij} = 1/i$ for $j = i + 1$ and $i \ge 2$). Answer the same questions as in parts ($a_1$) and ($a_2$) of this problem.

(c) Take $a = 2$ in the definition (1) of the chain and answer the same questions as in parts ($a_1$) and ($a_2$) of this problem.

Solution: For all the three cases, we have $f_{11}(1) = 0$, so below $n \ge 2$.

(a)

$$f_{11}(n) = 1\Big(1 - \frac{1}{2}\Big)\Big(1 - \frac{1}{3}\Big) \dots \Big(1 - \frac{1}{n-1}\Big)\frac{1}{n} = \frac{1}{n(n-1)},$$

and the probability of ever returning to 1 is

$$f_{11} = \sum_n f_{11}n = \sum_{n \ge 2} \frac{1}{n(n-1)} = 1.$$

Thus, state 1 is recurrent. The expected recurrence time $m_1 = \sum_n n f_{11}(n) = \sum_n \frac{1}{n-1} = \infty$, so the state (and the chain) is null recurrent.

By the main ergodic theorem for Markov chains, the limiting distribution does not exist. The proportion of time spent is state 1 in the long run is 0.

(b) A similar calculation now gives $f_{11}(n) = 1 \cdot \frac{1}{2} \cdots \frac{1}{n-1} \cdot \frac{n-1}{n} = \frac{1}{(n-1)!}\frac{n-1}{n}$ and $\sum_n f_{11}(n) = 1$. Moreover $\sum_{n \ge 2} n f_{11}(n) = \sum_{n \ge 2} \frac{1}{(n-2)!} = e$. In this case state 1 is positive recurrent, and $\pi_1 = \frac{1}{e}$.

(c) $f_{11}(n) = 1 \cdot \big(1 - \frac{1}{2^2}\big)\big(1 - \frac{1}{3^2}\big) \dots \big(1 - \frac{1}{(n-1)^2}\big)\frac{1}{n^2} = 1 \cdot \big(\frac{1}{2} \cdot \frac{3}{2}\big)\big(\frac{2}{3} \cdot \frac{4}{3}\big)\big(\frac{3}{4} \cdot \frac{5}{4}\big) \cdots \big(\frac{n-2}{n-1} \cdot \frac{n}{n-1}\big) \cdot \frac{1}{n^2} = \frac{1}{2n(n-1)}$. Then $\sum_{n \ge 2} f_{11}(n) = \frac{1}{2}\sum_{n \ge 2} \frac{1}{n(n-1)} = \frac{1}{2} < 1$, so state 1 is transient. There is no limiting distribution of the chain.

Problem 4.

(a) Given the generating function $g(z)$ of a random variable $Y$ supported on $\mathbb{N}_0$, the expectation $EY = g'(1)$. Express the variance $\mathrm{Var}(Y)$ using $g(z)$ (and its derivatives).

(b) Let $Z$ be the offspring random variable (the random number of children) in a branching process, and let $G(z)$ be the generating function of the distribution of $Z$. Suppose that the initial size of the population is $X_0 = 1$ and find the variance $\mathrm{Var}(X_n)$ of the population size $X_n$ in the $n$th generation. Assume that $EZ = \mu, \mathrm{Var}(Z) = \sigma^2$, where $Z$ is the RV representing the offspring distribution, and express your answer using only $\mu$, $\sigma^2$, and $n$.

Solution:

(a) $EY = g'(1)$ and $g''(1) = EY^2 - EY$, so $\mathrm{Var}(Y) = g''(1) + g'(1) - g'(1)^2$.

(b) Let $\mu = EZ = G'(1)$ and let $m_n = EX_n$. Let $G_n(z) = G(G(\ldots G(z))\ldots)$ ($n$ times) be the generating function of the distribution of $X_n$. We compute $m_n = G_n(z)'|_{z=1} = (G_{n-1}(G(z)))'|_{z=1} = G'_{n-1}(G(z))G'(z)|_{z=1} = m_{n-1}G'(1) = \mu m_{n-1} = \cdots = \mu^n$. We already know this from HW3.

Differentiating $G_n(z)$ twice, we obtain:

$$G''_n(z) = (G'_{n-1}(G(z))G'(z))' = G''_{n-1}(G(z))G'(z)^2 + G'_{n-1}(G(z))G''(z),$$

and

$$G''_n(1) = G''_{n-1}(1)(G(1)')^2 + G'_{n-1}(1)G''(1).$$

If $\mu = 1$, we obtain

$$\mathrm{Var}\,(X_n) = \sigma^2 + G^{(n-1)}(1)'' = \cdots = n\sigma^2.$$

if $\mu \neq 1$, then rewriting the above using part (a), we find $\mathrm{Var}\,(X_n) = \mu^2 \mathrm{Var}\,(X_{n-1}) + \mu^{n-1}\sigma^2$. Iterating this, we find

$$\mathrm{Var}\,(X_n) = \sigma^2(\mu^{n-1} + \mu^n + \cdots + \mu^{2(n-1)}) = \frac{\sigma^2\mu^{n-1}(\mu^n - 1)}{\mu - 1}.$$

Problem 5.

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $P(X_k = 1) = p = 1 - P(X_k = -1)$ for all $k$. Parts (a)-(c) below define sequences of RVs $(Y_n), (Z_n), (W_n)$ obtained from the sequence $(X_n)$. For each of these three sequences, answer the question whether it forms a (first-order homogeneous) Markov chain.

(a) $Y_n = X_n X_{n+1}, n \geq 1$;

(b) $Z_n = \frac{1}{2}(X_{n+1} - X_n), n \geq 1$;

(c) $W_n = \left|\sum_{k=1}^n X_k\right|, n \geq 1$.

Solution:

(a) $(Y_n)_n$ does not form a Markov chain. Indeed,

$$P(Y_3 = 1|Y_2 = 1) = \frac{P(Y_3 = 1, Y_2 = 1)}{P(Y_2 = 1)} = \frac{P(\{+++, ---\})}{P(\{++, --\})} = \frac{p^3 + (1-p)^3}{p^2 + (1-p)^2}.$$

$$P(Y_3 = 1|Y_2 = 1, Y_1 = 1) = \frac{P\{++++, ----\}}{P(\{++, --\})} = \frac{p^3 + (1-p)^4}{p^3 + (1-p)^3},$$

giving different values unless $p = 1/2, 0, 1$.

(b) $(Z_n)$ does not form a Markov chain. Indeed,

$$P(Z_3|Z_2 = 0, Z_1 = 1) = \frac{p(1-p)^3}{p^2(1-p)} = p$$

$$P(Z_3 = 0|Z_2 = 0, Z_1 = -1) = \frac{p(1-p)^3}{p(1-p)^2} = 1 - p,$$

giving different values unless $p = 1/2, 0, 1$.

(c) $(W_n)$ forms a Markov chain with the state space $S = \mathbb{N}_0$. Since $P_{01} = 1$, it suffices to analyze the process starting at 0 and until the next revisit of 0 because then the evolution is repeated exactly as before. So let us say that $W_0 = 0$, right before the start of the process. Note that between the two visits to 0, the sum $T_n := \sum_{n\geq 1} X_n$ does not change the sign, staying either in the positive or in the negative all the time. If is is in the positive, then the sum $T_n$ increases with probability $p$ and decreases with probability $q$, and $W_n$ does exactly the same. If it is in the negative, then $T_n$ increases with probability $q$ and decreases with probability $p$, and $W_n$ does the opposite. Moreover, $P(X_1 = 1) = p$ and $P(X_1 = -1) = q$, and the sign of $\sum_{n\geq 1} X_n$ stays fixed after that, determining the evolution of the process. Either way, these probabilities do not depend on the history given the current value of $W_n$.

The above argument suffices for an intuitive explanation. To give a proof, let us compute the transition probabilities of the Markov chain. If the process is in state $x$ after $n$ steps, then either $\frac{n+x}{2}$ values $X_k$ are $+1$

and $\frac{n-x}{2}$ values $X_k$ are $-1$ (the first case, $\sum_{k=1}^n X_k > 0$) or the opposite (the second case, $\sum_{k=1}^n X_k < 0$). Writing the probability for the first case,

$$M_n^+ := P(\sum_{k=1}^n X_n = x | X_n = x_n, \ldots, X_1 = x_1) = \frac{p^{\frac{n+x}{2}} q^{\frac{n-x}{2}}}{p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} + p^{\frac{n-x}{2}} q^{\frac{n+x}{2}}} = \frac{p^x}{p^x + q^x},$$

we observe that it does not depend on the earlier history given the value at time $n$, and also does not depend on $n$. A similar expression arises for the second case, namely, $M_n^- = \frac{q^x}{p^x+q^x}$, and the final answer is their (weighted) sum:

$$P(W_{n+1} = x_n + 1 | W_n = x_n, \ldots, W_1 = x_1) = pM^+ + qM^- = \frac{p^{x_n+1} + q^{x_n+1}}{p^{x_n} + q^{x_n}}.$$

Thus, the sequence $(W_n)$ forms a Markov chains, and its transition probabilities have the form

$$P_{ij} = \begin{cases} \frac{p^j+q^j}{p^i+q^i} & \text{if } j = i+1 \\ 1 - \frac{p^j+q^j}{p^i+q^i} & \text{if } j = i-1 \\ 0 & \text{o/w.} \end{cases}$$