Electrostatic energy

Consider a charge $q_1 \Rightarrow \Phi = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\underset{\sim}{x}-\underset{\sim}{x}_1|}$

Move another charge from ∞ to $\underset{\sim}{x}_2$.

The potential energy is

$$W = q_2 \Phi(\underset{\sim}{x}_2) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\underset{\sim}{x}_2-\underset{\sim}{x}_1|}$$

Now

$$\Phi(\underset{\sim}{x}) = \frac{q_1}{4\pi\epsilon_0|\underset{\sim}{x}-\underset{\sim}{x}_1|} + \frac{q_2}{4\pi\epsilon_0|\underset{\sim}{x}-\underset{\sim}{x}_2|}$$

Bringing in a charge q_3,

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{|\underset{\sim}{x}_2-\underset{\sim}{x}_1|} + \frac{q_1 q_3}{|\underset{\sim}{x}_3-\underset{\sim}{x}_1|} + \frac{q_2 q_3}{|\underset{\sim}{x}_3-\underset{\sim}{x}_2|}\right]$$

For n charges

$$W = \sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{|\underset{\sim}{x}_j-\underset{\sim}{x}_i|}\frac{1}{4\pi\epsilon_0}$$

or

$$W = \frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}\frac{q_i q_j}{|\underset{\sim}{x}_j-\underset{\sim}{x}_i|}\frac{1}{4\pi\epsilon_0}$$

where $i \neq j$ (no self energy)

For continuous charge distributions

$$W = \frac{1}{2} \int d\underline{x}\, d\underline{x}' \frac{\rho(\underline{x})\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} \frac{1}{4\pi\epsilon_0}$$

$$= \frac{1}{2} \int d\underline{x}\, \rho(\underline{x})\, \varphi(\underline{x})$$

$$= -\frac{1}{2}\epsilon_0 \int d\underline{x}\, \varphi(\underline{x})\, \nabla^2\varphi$$

Integrate by parts and assume $\underline{E} \to 0$ at ∞.

$$W = \frac{1}{2}\epsilon_0 \int d\underline{x}\, |\nabla\varphi|^2$$

$$W = \frac{1}{2}\epsilon_0 \int d\underline{x}\, |\underline{E}|^2$$

Thus,

$$w = \frac{1}{2}\epsilon_0 |\underline{E}|^2$$

is the energy density in an electrostatic field.

Note that the energy density is positive definite while the expression for discrete charges is not.

The expression

$$w = \frac{1}{2}\epsilon_0 |\mathbf{E}|^2$$

contains the self field.

Consider two point charges

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left[q_1 \frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|^3} + q_2 \frac{\mathbf{x}-\mathbf{x}_2}{|\mathbf{x}-\mathbf{x}_2|^3} \right]$$

$$w = \frac{1}{2}\epsilon_0 \left(\frac{1}{4\pi\epsilon_0}\right)^2 \left[\frac{q_1^2}{|\mathbf{x}-\mathbf{x}_1|^4} + \frac{q_2^2}{|\mathbf{x}-\mathbf{x}_2|^4} + 2 q_1 q_2 \frac{(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3 \, |\mathbf{x}-\mathbf{x}_2|^3} \right.$$

The first two terms involve self energy
$\Longrightarrow$ diverge when integrated over volume

Can integrate over the third term

$$W = \frac{q_1 q_2}{(4\pi)^2 \epsilon_0} \int d\mathbf{x} \; \underbrace{\frac{(\mathbf{x}-\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|^3}}_{\nabla \frac{1}{|\mathbf{x}-\mathbf{x}_1|}} \cdot \underbrace{\frac{(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_2|^3}}_{\nabla \frac{1}{|\mathbf{x}-\mathbf{x}_2|}}$$

Integrate by parts

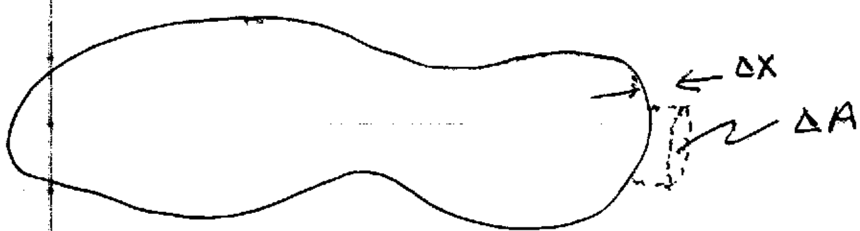
$$W = -\frac{q_1 q_2}{(4\pi)^2 \epsilon_0} \int d\underset{\sim}{x} \, \frac{1}{|\underset{\sim}{x}_2 - \underset{\sim}{x}_2|} \underbrace{\nabla^2 \frac{1}{|\underset{\sim}{x}_2 - \underset{\sim}{x}_1|}}_{-4\pi\,\delta(\underset{\sim}{x}_2 - \underset{\sim}{x}_1)}$$

$$= \frac{1}{4\pi\epsilon_0} q_1 q_2 \frac{1}{|\underset{\sim}{x}_2 - \underset{\sim}{x}_1|}$$

The self energy does not change as charges are assembled so, as long as the self field contributions don't diverge, the expression for the electric field energy density is ok.

Force on a conducting surface

Two approaches: energy change from virtual displacement and direct force calculation.



① Local energy density

$$\omega = \tfrac{1}{2}\epsilon_0 |\underline{E}|^2 \quad , \quad \underline{E} = \frac{\sigma}{\epsilon_0}$$

$$\omega = \frac{\sigma^2}{2\epsilon_0}$$

Consider a small displacement Δx of the conducting surface (virtual)

$\Longrightarrow$ change in energy

$$\Delta W = -\frac{\epsilon_0}{2}|\underline{E}|^2 \Delta a \Delta x$$

$$= -\frac{\sigma^2}{2\epsilon_0}\Delta a \Delta x$$

$$F = -\frac{\partial W}{\partial x} = \sigma^2 \Delta a \frac{1}{2\epsilon_0}$$

f = force per unit area

$$= \frac{\sigma^2}{2\epsilon_0}$$

$\Rightarrow$ force is outward from surface since displacement eliminates $\underline{E}$ within the displaced volume.

(2) Direct force calculation

$$f = \sigma E_{ext}$$

E_{ext} must exclude E_{self} from local charge density σ

$$E_{self} = \frac{\sigma}{2\epsilon_0}$$

$$E_n = E_{self} + E_{ext} = \frac{\sigma}{\epsilon_0}$$

$$\Rightarrow E_{ext} = \frac{\sigma}{2\epsilon_0}$$

$$f = \sigma^2 \frac{1}{2\epsilon_0}$$ $\Rightarrow$ same as energy argument

$\Rightarrow$ When calculating forces directly, always eliminate the self force $\Rightarrow$ nothing can accelerate itself.

Forces between charged objects

The force between charged objects can be calculated from Coulomb's law. However, for finite objects it is easier to derive the force from the stored energy. Consider the energy from a distribution of charges,

$$W = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \frac{1}{4\pi\epsilon_0}$$

Calculate the force on charge p,

$$\mathbf{F}_p = -\frac{\partial W}{\partial \mathbf{x}_p} = \sum_{i \neq p} \frac{q_i q_p}{|\mathbf{x}_p - \mathbf{x}_i|^3} \frac{(\mathbf{x}_p - \mathbf{x}_i)}{4\pi\epsilon_0}$$

The $\frac{1}{2}$ goes away because either $i = p$ or $j = p$.

$$\mathbf{F}_p = q_p \mathbf{E}(\mathbf{x}_p)$$

$\Rightarrow$ same answer from energy argument or direct force.

Can also calculate the force on a distribution of charge.

<u>Example</u> A conductor in an electric <u>field</u> $\underset{\sim}{E}_{ext}$

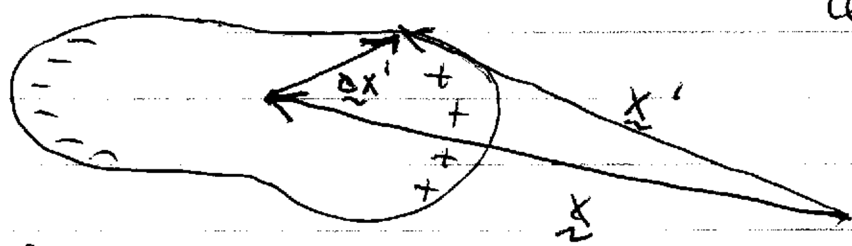
$\Rightarrow$ no net charge on conductor

$\Rightarrow$ object small compared with scale length over which $\underset{\sim}{E}_{ext}$ varies.

$\underset{\sim}{E}_{ext}$ $\longrightarrow$

$$\Phi = \Phi_{ext} + \Phi_{int}$$

$$\underset{\sim}{x}' = \underset{\sim}{x} + \Delta\underset{\sim}{x}'$$



<u>Stored energy</u>

$$W = \int ds'\,\sigma(\underset{\sim}{x}')\,\Phi_{ext}(\underset{\sim}{x}')$$ } work against external field to move charge from infinity

$$+ \frac{1}{2}\int ds'\,\sigma(\underset{\sim}{x}')\,\underbrace{\Phi_{int}(\underset{\sim}{x}')}_{\underbrace{\Phi_{int}+\Phi_{ext}}_{\Phi}-\Phi_{ext}}$$ } work to assemble charge

$$= \frac{1}{2}\int ds'\,\sigma(\underset{\sim}{x}')\,\Phi_{ext}(\underset{\sim}{x}')$$

$$+ \frac{1}{2}\underbrace{\int ds'\,\sigma(\underset{\sim}{x}')\,\Phi(\underset{\sim}{x}')}_{\Phi_{cond}\underbrace{\int ds'\,\sigma(\underset{\sim}{x}')}_{0}}$$

$\Rightarrow \Phi$ const. on conductor

$$W = \frac{1}{2}\int ds'\,\sigma(\underset{\sim}{x}')\,\varphi_{ext}(\underset{\sim}{x}')$$

$$\underset{\sim}{E} = -\nabla\varphi$$

$$\varphi(\underset{\sim}{x}') - \varphi(\underset{\sim}{x}) = -\int_{\underset{\sim}{x}}^{\underset{\sim}{x}'} d\underset{\sim}{x}''\cdot\underset{\sim}{E}(\underset{\sim}{x}'')$$

$$\simeq -\underset{\sim}{E}(\underset{\sim}{x})\cdot(\underset{\sim}{x}'-\underset{\sim}{x})$$

$$W = \frac{1}{2}\int ds'\,\sigma(\underset{\sim}{x}')\left[\cancelto{0}{\varphi_{ext}(\underset{\sim}{x})} - \underset{\sim}{E}_{ext}(\underset{\sim}{x})\cdot\Delta\underset{\sim}{x}'\right]$$

since no
net charge

$$= -\frac{1}{2}\underbrace{\int ds'\,\sigma(\underset{\sim}{x}')\,\Delta\underset{\sim}{x}'}_{\underset{\sim}{P}\,=\,\text{dipole moment}}\cdot\underset{\sim}{E}_{ext}(\underset{\sim}{x})$$

$$W = -\frac{1}{2}\underset{\sim}{P}\cdot\underset{\sim}{E}\,,\qquad \underset{\sim}{P} = \int ds'\,\sigma(\underset{\sim}{x}')\,\Delta\underset{\sim}{x}'$$

$\Rightarrow$ only valid in a system in which $\underset{\sim}{P}$ is induced by $\underset{\sim}{E}_{ext}$. Permanent dipole has no 1/2 in front.

Force acting on conductor ?

$$\underset{\sim}{F}_{tot} = \int ds'\,\sigma(\underset{\sim}{x}')\,\underset{\sim}{E}_{ext}(\underset{\sim}{x}')$$

$\Rightarrow$ $\underset{\sim}{E}_{int}$ can not produce a net force

$$= \int ds'\,\sigma(\underset{\sim}{x}')\,\underbrace{\underset{\sim}{E}_{ext}(\underset{\sim}{x} + \Delta\underset{\sim}{x}')}_{\underset{\sim}{E}_{ext}(\underset{\sim}{x}) + \Delta\underset{\sim}{x}'\cdot\nabla\underset{\sim}{E}_{ext}}$$

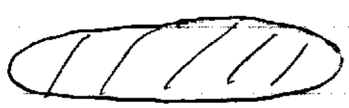
$$= \cancelto{0}{\int ds'\,\sigma(\underset{\sim}{x}')}\;\underset{\sim}{E}_{ext}(\underset{\sim}{x}) + \int ds'\,\sigma(\underset{\sim}{x}')\,\Delta\underset{\sim}{x}'\cdot\nabla\underset{\sim}{E}_{ext}$$

no net charge

$$= \underset{\sim}{P}\cdot\nabla\underset{\sim}{E}_{ext}$$

For a conductor with symmetry

$\longrightarrow \underset{\sim}{E}_{ext}$



$$\underset{\sim}{P} = \kappa\,\underset{\sim}{E}_{ext}$$

κ = a constant

$\Rightarrow$ depends on geometry

$$\underset{\sim}{F}_{tot} = \kappa\,\underset{\sim}{E}_{ext}\cdot\nabla\underset{\sim}{E}_{ext}$$

$$\underset{\sim}{E}\times(\nabla\times\underset{\sim}{E}) = 0 = \underbrace{(\nabla\underset{\sim}{E})\cdot\underset{\sim}{E}}_{(\nabla E_i)E_i = \frac{1}{2}\nabla E_i^2 = \frac{1}{2}\nabla E^2} - \underset{\sim}{E}\cdot\nabla\underset{\sim}{E}$$

$\Rightarrow \quad \underset{\sim}{E} \cdot \nabla \underset{\sim}{E} = \frac{1}{2} \nabla E^2$

$\underset{\sim}{F}_{tot} = \frac{1}{2} \kappa \nabla E_{ext}^2$

$\Rightarrow$ force points in direction of larger E_{ext}^2

$\Rightarrow$ W is more negative in region of larger $\underset{\sim}{E}_{ext}$

Can also calculate force from W.

$$\underset{\sim}{F}_{tot} = - \frac{\partial}{\partial \underset{\sim}{x}} W = - \frac{\partial}{\partial \underset{\sim}{x}} \left(-\frac{1}{2} \underset{\sim}{P} \cdot \underset{\sim}{E}_{ext} \right)$$

$\Rightarrow$ note that $\underset{\sim}{P}(\underset{\sim}{x})$ also changes with $\underset{\sim}{x}$

$$\underset{\sim}{F}_{tot} = \frac{\partial}{\partial \underset{\sim}{x}} \frac{1}{2} \kappa |\underset{\sim}{E}_{ext}|^2$$

$\Rightarrow$ same as before

Capacitance

Consider an isolated conductor with a charge Q. The potential φ is given by

$$\varphi = \frac{Q}{C}$$

where C is the capacitance. If C is large, the conductor can store a lot of charge while changing its ~~point~~ potential by a small amount. Note that C depends only on the geometry of the conductor.

The capacitance of two conductors of equal and opposite charge is

$$C = \frac{Q}{\varphi_2 - \varphi_1}$$