MHD Turbulence and Dissipation

Magnetic field are measured throughout the universe :
planetary bodies, stars, the heliosphere, the galaxy, at supernova shocks etc.
⇒ believed to play a critical role in the accretion of matter into compact objects, star formation
⇒ angular momentum transport

Is the magnetic field primordial or locally generated through a dynamo?
⇒ planetary and stellar objects have dynamo action ⇒ field reversals are clear evidence for active dynamos
⇒ accretion discs, supernova shocks ⇒ dynamo
⇒ galactic scale not clear.

Generation mechanism is the dynamo
⇒ twisting action by convective flows leads to amplification of the magnetic field

Dissipation Mechanism — magnetic reconnection release of energy into high speed flows and energetic particles.

www.scholarpedia.org/article/MHD_turbulence.

MHD Turbulence

~~Neglect~~
Broadly important both within the heliosphere

⇒ solar wind ⇒ heating ⇒ $\tilde{V}^{-5}$ ~ f for $v > c_A$
⇒ corona ⇒ heating

Astrophysics — star formation
accretion

Ideal invariants ⇒ neglecting viscosity and resistivity.

3-D

$$E = \frac{1}{2}\int d\underline{x}\left(V^2 + B^2\right) = \text{energy}$$

$$K = \frac{1}{2}\int d\underline{x}\ \underline{B}\cdot\underline{V} = \text{cross helicity}$$

$$H = \frac{1}{2}\int d\underline{x}\ \underline{A}\cdot\underline{B} = \text{magnetic helicity}.$$

$$\underline{B} = \nabla \times \underline{A}$$

2-D

E, K and $\qquad \underline{B} = \hat{z}\times\nabla\psi$

$$A = \int d\underline{x}\ \frac{1}{2}\psi^2$$

⇒ note two quadratic invariants implies that
A has an inverse cascade

Focus on 3-D systems in incompressible limit

$\Rightarrow$ neglect magnetosonic and sound waves

An MHD system differs from fluid system because of magnetic tension which causes disturbances to have a finite frequency.

$$\omega^2 = \frac{(\underset{\sim}{k}\cdot\underset{\sim}{B}_0)^2}{4\pi\rho} = k_\parallel^2 C_{A0}^2$$

Note that even if the system has no mean magnetic field can treat $\underset{\sim}{B}_0$ as the locally averaged field defined by the scale length somewhat greater than scales of interest.

$\Rightarrow$ a local quantity.

$\Rightarrow$ cascade depends critically on B_0

Three important properties:

(1) wave packets propagating in one direction are exact solutions of MHD eqns $\Rightarrow$ Define Elsässer variables

$$z^\pm = \tilde{v} \pm \tilde{b}$$

where write $\tilde{b}$ in velocity units

$$\tilde{b} = \frac{\tilde{B}}{\sqrt{4\pi\rho}}$$

$\Rightarrow$ MHD eqns

$$\frac{\partial}{\partial t} \underset{\sim}{z}^{\pm} \mp \underset{\sim}{v}_{A0} \cdot \nabla \underset{\sim}{z}^{\pm} = - \underset{\sim}{z}^{\mp} \cdot \nabla \underset{\sim}{z}^{\pm} - \nabla P$$

$$\nabla^2 P = - \nabla\nabla : \underset{\sim}{z}^{+} \underset{\sim}{z}^{-}$$

If z^+ or z^- is zero, the other is an exact solution

(2) Interaction between oppositely propagating disturbances is what drives the turbulent cascade.

(3) Disturbances z^+, z^- to not exchange energy when they interact

$\Rightarrow$ can write conservation laws in terms of $z^{\pm}$

$$E = \frac{1}{4} \int d\underset{\sim}{x} \left(z^{+2} + z^{-2} \right)$$

$$K = \int d\underset{\sim}{x}\, \underset{\sim}{v} \cdot \underset{\sim}{B} = \frac{1}{4} \int d\underset{\sim}{x} \left(z^{+2} - z^{-2} \right)$$

$\Rightarrow$ can't change energy while also preserving K.

$\Rightarrow$ If a system initially has ~~[illegible]~~ very little energy in z^-, this will not change unless some other process intervenes

MHD turbulence cascades preferentially in the direction $\perp$ to $\underline{B}_0$ so that the turbulent spectra in MHD are anisotropic

$\Rightarrow$ more rapid fall-off in $k_{\parallel}$ direction than in $k_{\perp}$.

To understand why the cascade along $k_{\parallel}$ is inhibited look ~~at~~ at three wave interactions

$\Rightarrow$ $\underline{k}_1, \underline{k}_2$ generating $\underline{k}_3$

Must have ω and $\underline{k}$ matching.

$$\underline{k}_1 + \underline{k}_2 = \underline{k}_3$$

$$\omega_1 - \omega_2 = \pm \omega_3$$

where $\omega_1 = k_{\parallel 1} C_{A0}$, $\omega_2 = k_{\parallel 2} C_{A0}$, $\omega_3 = k_{\parallel 3} C_{A0}$

Note we have taken $\underline{k}_2$ to have a negative frequency, which is required so that $z_{1,2}$ are propagating in opposite directions. Looking at $k_{\parallel}$ matching and ω matching,

$$k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3}$$

$$k_{\parallel 1} - k_{\parallel 2} = \pm k_{\parallel 3}$$

Adding and subtracting, we obtain

$$k_{\parallel 1} = \frac{1}{2}\left(k_{\parallel 3} \pm k_{\parallel 3}\right) = \begin{pmatrix} k_{\parallel 3} \\ 0 \end{pmatrix}$$

$$k_{\parallel 2} = \frac{1}{2}\left(k_{\parallel 3} \mp k_{\parallel 3}\right) = \begin{pmatrix} 0 \\ k_{\parallel 3} \end{pmatrix}$$

Thus, either $k_{\parallel 1}$ or $k_{\parallel 2}$ is zero.

$\Rightarrow$ no coupling of energy to larger $k_{\parallel 3}$

Shebalin, Matthaeus, Montgomery '83

$\Rightarrow$ no parallel cascade

Critical Balance hypothesis

A key parameter in MHD cascade physics is the ~~ratio~~ relation between the linear frequency and the nonlinearity

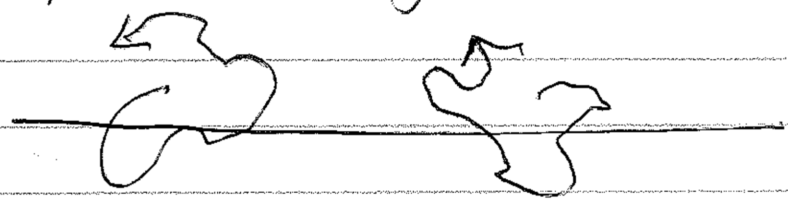
$$\omega \sim k_{\parallel} C_{A0} \quad \text{frequency}$$

$$\sim k_{\perp} V_{k_\perp} \quad \text{nonlinear rate}$$

It has been suggested that the parallel wavelength will adjust so that at any transverse scale have a balance so that

$$k_{\parallel} C_{A0} \sim k_{\perp} V_{k_\perp}$$

The argument is that if $k_\parallel$ is too small the turbulence at two locations will evolve independently, and will therefore generate a $k_\parallel$,



$k_\parallel$ increases until propagation along $\underline{B}_0$ causes the two locations to correlate their transverse motion.

~~Spectral~~ Inertial Range Spectra

Look at the cascade just as in hydrodynamic case. Consider a discrete set of scales $l_n \sim k_{\perp n}^{-1}$. For magnetic fields in velocity units

$$V_n \sim b_n$$

Energy transfer rate

$$\epsilon = \frac{E_n}{\tau_n} \sim V_n^2 \frac{V_n}{l_n} \sim \frac{V_n^3}{l_n}$$

$$V_n \sim (\epsilon l_n)^{\frac{1}{3}}$$

$$E_n \sim V_n^2 \sim \epsilon^{2/3} l_n^{2/3} \sim E(k_n)\,\Delta k_{\perp n} \Rightarrow \text{1-D spectral energy}$$

$$\sim E(k_{\perp n})\,\frac{1}{l_n}$$

$$\Longrightarrow \quad E(k_{\perp n}) \sim \epsilon^{2/3} (k_\perp)^{-5/3}$$

Now use critical balance to ~~s~~ obtain parallel energy spectrum $k_\parallel = k_\parallel(k_\perp)$

$$k_\parallel C_{A0} \sim k_{\perp n} V_n \sim \frac{V_n}{l_n} \sim \epsilon^{1/3} \frac{1}{l_n^{2/3}}$$

$$k_\parallel \sim \epsilon^{1/3} k_\perp^{+2/3}$$

$$\boxed{k_\parallel \sim \cancel{\frac{k}{k_\perp}}\, k_\perp^{2/3}}$$

Goldreich Sridhar '95

$\Rightarrow$ faster fall-off in the $k_\parallel$ direction