Hwk # 1 Solutions

1. Nonlinear sound wave eqns

continuity $$\frac{\partial}{\partial t} n + \frac{\partial}{\partial x} nv = 0$$

momentum $$n\left(\frac{\partial v}{\partial t} + v\frac{\partial}{\partial x} v\right) = -\frac{T}{m_i}\frac{\partial n}{\partial x} + n_0\nu\frac{\partial^2 v}{\partial x^2}$$

To normalize the equations first look at linear dispersion.

$$\omega(\omega + i\nu k^2) = k^2 c_s^2$$

$\Rightarrow$ normalize so that propagation and dissipation are comparable rates

$$\Rightarrow k c_s \sim \nu k^2$$

$$\Rightarrow k \sim \frac{c_s}{\nu} \Rightarrow L \sim \frac{\nu}{c_s}$$

$$\tau \sim \frac{1}{kc_s} \sim \frac{\nu}{c_s^2}$$

$$v \sim c_s$$

$$n \sim n_0$$

$\Rightarrow$ normalized eqns

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n v) = 0$$

$$n\left(\frac{\partial v}{\partial t} + v \frac{\partial}{\partial x} v\right) = -\frac{\partial n}{\partial x} + \frac{\partial^2}{\partial x^2} v$$

② Jump to moving frame (velocity 1)

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial t} n - \frac{\partial}{\partial x} n + \frac{\partial}{\partial x} n v = 0$$

$$n\left(\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} + v \frac{\partial}{\partial x} v\right) = -\frac{\partial n}{\partial x} + \frac{\partial^2}{\partial x^2} v$$

(suppress primes on x, t)

$\Rightarrow$ assume weak nonlinearity balanced by dissipation

$$\frac{\partial}{\partial t} \sim k v \sim k^2$$

$$\Rightarrow k \sim v$$

$$\Rightarrow \text{take } k \sim v \sim \epsilon \ll 1$$

$$\frac{\partial}{\partial t} \sim k v \sim \epsilon^2, \quad \frac{\partial}{\partial x} \sim \epsilon$$

3 Take $n = 1 + \epsilon n_1 + \epsilon^2 n_2 + \cdots$

$$V = \epsilon V_1 + \epsilon^2 V_2 + \cdots$$

lowest order

$$\left.\begin{aligned} -\frac{\partial n_1}{\partial x} + \frac{\partial}{\partial x} V_1 &= 0 \\ -\frac{\partial V_1}{\partial x} &= -\frac{\partial n_1}{\partial x} \end{aligned}\right\} \quad V_1 = n_1$$

<u>next order</u>

$$\frac{\partial n_1}{\partial t} - \frac{\partial}{\partial x} n_2 + \frac{\partial}{\partial x} V_2 + \frac{\partial}{\partial x} n_1 V_1 = 0$$

$$\frac{\partial V_1}{\partial t} - \frac{\partial V_2}{\partial x} - n_1 \frac{\partial V_1}{\partial x} + V_1 \frac{\partial}{\partial x} V_1 = -\frac{\partial n_2}{\partial x} + \frac{\partial^2}{\partial x^2} V_1$$

$\Rightarrow$ add eqns $\Rightarrow$ eliminates n_2, V_2

$$2\frac{\partial n_1}{\partial t} + \frac{\partial}{\partial x} n_1^2 - \cancel{n_1 \frac{\partial n_1}{\partial x}} + \cancel{n_1 \frac{\partial}{\partial x} n_1} = -\frac{\partial^2}{\partial x^2} n_1$$

$$\boxed{\frac{\partial n_1}{\partial t} + n_1 \frac{\partial}{\partial x} n_1 + \frac{1}{2} \frac{\partial^2}{\partial x^2} n_1 = 0}$$

(4)

Look for steady state solutions with velocity V_0

$$\cancel{n} \quad n(x,t) = n(x - v_0 t)$$

$$\frac{\partial n}{\partial t} = -V_0 \frac{\partial}{\partial x} n$$

$$-V_0 \frac{\partial}{\partial x} n + \frac{1}{2} \frac{\partial}{\partial x} n^2 - \frac{1}{2} \frac{\partial^2}{\partial x^2} n = 0$$

$\Rightarrow$ take first integral

$$-V_0 n + \frac{1}{2} n^2 - \frac{1}{2} \frac{\partial n}{\partial x} = \text{const.} = -a \frac{1}{2}$$

On either side of the shock, $\frac{\partial n}{\partial x} \to 0$

$$n^2 - 2V_0 n + a = 0$$

$$n = \frac{\cancel{2}V_0 \pm \sqrt{\cancel{4}V_0^2 - \cancel{4}a}}{\cancel{2}}$$

$$= V_0 \pm \sqrt{V_0^2 - a}$$

$$n = n^{\pm}$$

$$\boxed{\frac{n^+ + n^-}{2} = V_0} \qquad \boxed{n^- = n^+ - 2\sqrt{V_0^2 - a}}$$

valid for small V_0

$$(n-n^+)(n-n^-) = \frac{dn}{dx}$$

$$\int \frac{dn}{(n-n^+)(n-n^-)} = \int dx$$

$$\int dn \left[\frac{1}{(-n+n^+)} + \frac{1}{(n-n^-)} \right] = -\int dx \, (n^+-n^-)$$

$$\ln \frac{n-n^-}{n_+-n} = -(n^+-n^-)\, x \qquad \Delta n \equiv n^+-n^-$$

$$\frac{n-n^-}{n_+-n} = e^{-\Delta n x}$$

$$(n-n^-) = e^{-\Delta n x}(n_+-n)$$

$$n(1+e^{-\Delta n x}) = n^- + e^{-\Delta n x} n^+$$

$$\boxed{n = \frac{n^- + n^+ e^{-\Delta n x}}{1+e^{-\Delta n x}}}$$

$$x \to -\infty \qquad n \to n^+$$

$$x \to +\infty \qquad n \to n^-$$

$$\boxed{\Delta x = \frac{1}{\Delta n}\frac{v}{c_s}}$$