Physics 761 Hwk #4 Solutions

(1)
$$Z(\varsigma) = \int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{s-\varsigma}$$

$$\frac{dZ}{d\varsigma} = \int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{(s-\varsigma)^2}$$ integrate by parts

$$= -\int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}} \left(-\frac{1}{(s-\varsigma)}\right)\left(-2s e^{-s^2}\right)$$

$$= -2 \int \frac{ds}{\sqrt{\pi}} \frac{s-\varsigma+\varsigma}{s-\varsigma} e^{-s^2}$$

$$= -2 - 2\varsigma \int \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{s-\varsigma}$$

$$= -2\left(1 + \varsigma Z(\varsigma)\right)$$

(2) Consider stationary ions and electrons drifting with velocity $\underset{\sim}{u}_e = u\hat{z}$.
Take $\underset{\sim}{k}$ parallel to $\underset{\sim}{u}_e$ $\Rightarrow$ 1-D problem

$$\frac{\partial}{\partial t} f + v_z \frac{\partial}{\partial z} f - \frac{q}{m} \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial v_z} f = 0$$

Linearize equation with
$$f = f_0 + \mathrm{Re}\left(\hat{f} e^{ikz - i\omega t}\right)$$

$$(-i\omega + ikv_z)\hat{f} - \frac{q}{m} ik\,\hat{\varphi}\,\frac{\partial}{\partial v_z} f_0 = 0$$

$$\hat{f} = \frac{\frac{q}{m}\hat{\varphi}\,\frac{\partial}{\partial v_z} f_0}{v_z - \frac{\omega}{k}}$$

$$\nabla \cdot \underset{\sim}{E} = 4\pi e\,(n_i - n_e)$$

$$k^2 \hat{\varphi} = 4\pi e(\hat{n}_i - \hat{n}_e) \qquad v_p = \frac{\omega}{k}$$

$$k^2 = 4\pi \sum_{i,e} \frac{q^2}{m} \int dv_z \frac{1}{v_z - v_p} \frac{\partial}{\partial v_z} f_0$$

$$= 4\pi \sum_{i,e} \frac{q^2}{m} \int dv_z \frac{f_0}{(v_z - v_p)^2}$$

$$= 4\pi \sum_{i,e} \frac{q^2 n_0}{m} \int \frac{dv_z}{(2\pi T/m)^{\frac{1}{2}}} \frac{e^{-(v_z-u)^2/v_t^2}}{(v_z - v_p)^2}$$

$$= 4\pi \sum_{ie} \frac{q^2 n_0}{m v_t^2} \int \frac{ds}{\sqrt{\pi}} \frac{e^{-(s - u/v_t)^2}}{\left(s - \frac{\omega}{k v_t}\right)^2}$$

$$\frac{1}{2} m v_t^2 = T$$

$$k^2 = \frac{k_{Di}^2}{2} Z'\left(\frac{\omega}{k v_{ti}}\right) + \frac{k_{De}^2}{2} Z'\left(\frac{\omega - ku}{k v_{te}}\right)$$

$$\boxed{\epsilon(k,\omega) = 1 - \frac{k_{Di}^2}{2k^2} Z'\left(\frac{\omega}{k v_{ti}}\right) - \frac{k_{De}^2}{2k^2} Z'\left(\frac{\omega - ku}{k v_{te}}\right)}$$

(3) Take $T_e \gg T_i$.

Since $\omega_R \sim k c_s$, for electrons

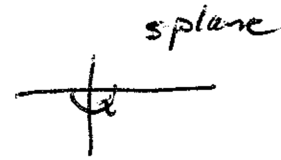
$$\frac{\omega - ku}{k v_{te}} \sim \frac{k(c_s - u)}{k v_{te}} \sim \frac{c_s - u}{v_{te}} \ll 1$$

since $u / v_{te} \ll 1$ and $c_s \sim \sqrt{\frac{m_e}{m_i}}\, v_{te}$.

$\Rightarrow$ evaluate Z' for electrons in the small argument limit.

$$Z'\left(\frac{\omega - ku}{k v_{te}}\right) = -2\left(1 + \frac{\omega - ku}{k v_{te}} Z(\;)\right)$$

$$Z\left(\frac{\omega - ku}{k v_{te}}\right) \simeq \int \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{s - \left(\frac{\omega - ku}{k v_{te}}\right)}$$



$$\simeq \frac{i\pi}{\sqrt{\pi}}$$

$$Z'\left(\frac{\omega - ku}{k v_{te}}\right) \simeq -2\left[1 + \frac{\omega - ku}{k v_{te}} i\sqrt{\pi}\right]$$

For ions

$$\frac{\omega}{k v_{ti}} \sim \frac{c_s}{v_{ti}} \sim \sqrt{\frac{T_e}{T_i}} \gg 1$$

$$Z'\left(\frac{\omega}{kv_{ti}}\right) = -2\left(1 + 2\,\frac{\omega}{kv_{ti}}\,Z\left(\frac{\omega}{kv_{ti}}\right)\right)$$

$$Z\left(\frac{\omega}{kv_{ti}}\right) = -\frac{1}{\xi}\left(1 + \frac{1}{2\xi^2}\right) + i\sqrt{\pi}\, e^{-\frac{\omega^2}{k^2 v_{ti}^2}}$$

$$Z'\left(\frac{\omega}{kv_{ti}}\right) \simeq -2\left(1 + \xi\left[\left(-\frac{1}{\xi}\right)\left(1+\frac{1}{2\xi^2}\right) + i\sqrt{\pi}\, e^{-\xi^2}\right]\right)$$

$$= -2\left(-\frac{1}{2\xi^2} + i\xi e^{-\xi^2}\sqrt{\pi}\right)$$

$$= \frac{1}{\xi^2} - 2i\sqrt{\pi}\,\xi e^{-\xi^2}$$

$$= \frac{k^2 v_i^2}{\omega^2} - 2i\sqrt{\pi}\,\frac{\omega}{kv_i}\, e^{-\omega^2/k^2 v_i^2}$$

$\Rightarrow$ dispersion relation

$$1 - \frac{k_{Di}^2}{2k^2}\left[\frac{k^2 v_i^2}{\omega^2} - 2i\sqrt{\pi}\,\frac{\omega}{kv_i}\, e^{-\omega^2/k^2 v_i^2}\right]$$

$$+ \frac{k_{De}^2}{\cancel{2}k^2}\left[1 + i\sqrt{\pi}\,\frac{\omega - ku}{kv_{te}}\right] = 0$$

$$1 + \frac{k_{De}^2}{k^2} - \frac{\omega_{pi}^2}{\omega^2} + i\sqrt{\pi}\,\frac{k_{De}^2}{k^2}\left[\frac{\omega-ku}{kv_{te}} + \frac{T_e}{T_i}\,\frac{\omega}{kv_{ti}}\, e^{-\frac{\omega^2}{k^2 v_{ti}^2}}\right] = 0$$

$$\omega_{pi}^2 = \frac{4\pi n e^2}{m_i}$$

Imaginary part small since

$$\frac{\omega - ku}{kV_{te}} \ll 1 \quad \text{and} \quad e^{-\frac{\omega^2}{k^2 V_{ti}^2}} \ll 1$$

Lowest order:

$$1 + \frac{k_{De}^2}{k^2} - \frac{\omega_{pi}^2}{\omega_0^2} = 0 \qquad \omega_0^2 = \frac{\omega_{pi}^2}{1 + k_{De}^2/k^2}$$

$$= \frac{k^2 c_s^2}{1 + k^2/k_{De}^2}$$

Next order:

$$\omega = \omega_0 + i\gamma$$

$$-\frac{\partial}{\partial \omega}\left(\frac{\omega_{pi}^2}{\omega^2}\right)\Bigg|_{\omega_0} i\gamma + i\sqrt{\pi}\,\frac{k_{De}^2}{k^2}\Big[\quad\Big]\Bigg|_{\omega=\omega_0} = 0$$

$$\gamma = -\sqrt{\pi}\,\frac{k_{De}^2}{k^2}\,\frac{\omega_0^3}{2\omega_{pi}^2}\left[\frac{\omega_0 - ku}{kV_{te}} + \frac{T_e}{T_i}\,\frac{\omega}{kV_{ti}}\,e^{-\omega^2/k^2 V_{ti}^2}\right]$$

For $T_i = 0$, instability for

$$\frac{\omega_0 - ku}{kV_{te}} < 0 \quad \text{or} \quad \frac{\cancel{k^2} c_s^2}{1 + k^2/k_{De}^2} < \cancel{k^2} u^2$$

$$\Rightarrow \boxed{u^2\left(1 + k^2/k_{De}^2\right) > c_s^2}$$