Homework #6 Solutions Physics 761

① Write the quasilinear eqns for ion acoustic instability with cold ions.

dispersion relation $\left(\frac{\omega}{kv_i} \gg 1, \ \frac{\omega}{kv_e} \ll 1 \right)$

$$\epsilon = 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{k_{De}^2}{k^2} - \frac{\omega_{pe}^2}{n_0 k^2} i\pi \frac{\partial f_0}{\partial v}\Big|_{v_p} = 0$$

$\Rightarrow$ don't assume electrons are Maxwellian since will evolve in time.

To lowest order ($\gamma_k \ll \omega_0$)

$$\omega_0 = \frac{kc_s}{(1 + k^2/k_{De}^2)^{\frac{1}{2}}} \qquad v_p = \frac{c_s}{(1 + k^2/k_{De}^2)^{\frac{1}{2}}}$$

First order:

$$i\gamma_k \frac{\partial}{\partial\omega}\left(-\frac{\omega_{pi}^2}{\omega^2}\right) - i\pi \frac{\omega_{pe}^2}{n_0 k^2} \frac{\partial f_0}{\partial v}\Big|_{v_p} = 0$$

$$\gamma_k = \pi \frac{\omega_0^3}{2\omega_{pi}^2} \frac{\omega_{pe}^2}{n_0 k^2} \frac{\partial f_0}{\partial v}\Big|_{v_p}$$

$\Rightarrow$ unstable for $0 < v_p < u$ at $t = 0$.

Electric field:

$$\frac{\partial}{\partial t}|\mathcal{E}_k|^2 = 2\gamma_k |\mathcal{E}_k|^2$$

Electrons — as beam plasma instability

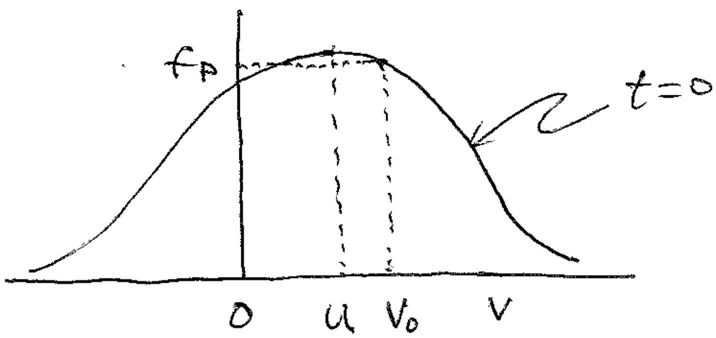
$$\frac{\partial f_0}{\partial t} - \frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_0 = 0 \qquad D = \frac{e^2}{m_e^2} \sum_k \frac{\gamma_k k^2 |\mathcal{E}_k|^2}{\gamma_k^2 + (\omega_k - kv)^2}$$

$$f_0(v, t=0) = n_0 \frac{e^{-(v-u)^2/v_{te}^2}}{\sqrt{\pi}\, v_{te}}$$

Electrons evolve with time but ions remain linear.

(2) Electrons will flatten their distribution in the region $0 < v < v_0$, where v_0 can not be longer than c_s, the max. phase velocity of the ion acoustic wave. (2)

First assume that u is small enough so a plateau can form that flattens the electrons completely in the region $v > 0$.



$$f_p = f_0(v_0, 0)$$

From electron number conservation

$$f_p v_0 = \int_0^{v_0} dv\, f_0(v,0)$$

$$v_0 \cancel{\frac{1}{\sqrt{\pi}}} \cancel{n_0} \left(\cancel{1} - \frac{(v_0-u)^2}{v_{te}^2}\right) = \cancel{\frac{1}{\sqrt{\pi}}} \cancel{n_0} \int_0^{v_0} dv \left[\cancel{1} - \frac{(v-u)^2}{v_{te}^2}\right]$$

$$v_0 (v_0 - u)^2 = \frac{(v_0-u)^3}{3} - \frac{(-u)^3}{3}$$

$$3 v_0 (v_0^2 - 2 v_0 u + \cancel{u^2}) = v_0^3 - 3 v_0^2 u + \cancel{3 v_0 u^2}$$

$$2 v_0^{\cancel{3}\,2} = 3 \cancel{v_0} u \quad \Rightarrow \boxed{v_0 = \frac{3u}{2}}$$

$\Rightarrow$ requires $\boxed{\frac{3u}{2} < c_s}$

$\Rightarrow$ no waves beyond c_s

3 Now calculate the wave spectrum.

As in the beam plasma instability need to write $\partial f_0/\partial v$ in the diffusion equation in terms of the growth rate.

$$\epsilon = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{k^2 n_0}\int_{-\infty}^{\infty} dv \, \frac{\partial f_0/\partial v}{v - \frac{\omega}{k}}$$

$$= 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{k_{De}^2}{k^2} - \frac{\omega_{pe}^2}{k^2 n_0}\, i\pi \, \frac{\partial f_0}{\partial v}\Big|_{v_p}$$

$$\left[\frac{\partial}{\partial \omega}\left(-\frac{\omega_{pi}^2}{\omega^2}\right)\right]\Big|_{\omega_0} \cancel{i}\gamma - \frac{\omega_{pe}^2}{k^2 n_0}\cancel{i}\pi \, \frac{\partial f_0}{\partial v}\Big|_{v_p} = 0$$

$$\gamma_k = \frac{m_i}{m_e}\frac{\omega_0^3}{k^2}\frac{\pi}{2 n_0}\frac{\partial f_0}{\partial v}\Big|_{v_p}$$

Taking γ_k small and $\frac{\gamma_k}{\gamma_k^2 + |\omega_k - kv|^2} = \pi\,\delta(\omega_k - kv)$

$$\frac{\partial}{\partial t} f_0 - \frac{\partial}{\partial v}\pi \frac{e^2}{m_e^2}\sum_k \cancel{k}\,|E_k|^2\,\delta(\omega_k - kv)\frac{\partial}{\partial v} f_0 = 0$$

$$\frac{\partial}{\partial t} f_0 - \frac{\partial}{\partial v}\cancel{\pi}\frac{e^2}{m_e^{\cancel{2}}}\sum_k |E_k|^2\,\delta(\omega_k - kv)\,\frac{2 n_0}{\cancel{\pi}}\frac{k^2}{\omega_0^3}\frac{\cancel{m_e}}{m_i}\gamma_k = 0$$

$$\frac{\partial}{\partial t}\left(f_0 - \frac{\partial}{\partial v}\cancel{\frac{4\pi n_0 e^2}{m_i}}\,\omega_{pi}^2\frac{1}{m_e}\sum_k \frac{|E_k|^2}{4\pi}\,\delta()\,\cancel{k}\,k^2\frac{1}{\omega_0^3}\right) = 0$$

Integrating over time

$$f_p - \frac{\partial}{\partial v}\frac{1}{m_e}\sum_k k^2\frac{|E_k|^2}{4\pi}\frac{\omega_{pi}^2}{\omega_0^3}\delta() = f_0(v, t=0)$$

Integrate over velocity from 0 to v.

$$\frac{1}{m_e}\sum_k \frac{k^2|E_k|^2}{4\pi}\,\frac{\omega_{pi}^2}{\omega_0^3}\,\delta(\omega_k - kv) = \int_0^v dv\,(f_p - f_0)$$

$$\int_0^v dv\,(f_p - f_0) = \int_0^v dv\left[f_0\left(\tfrac{3}{2}u, 0\right) - f_0(v, 0)\right]$$

$$= \int_0^v dv\,\frac{n_0}{\sqrt{\pi}\,v_{te}}\left[\cancel{1} - \frac{1}{4}\frac{u^2}{v_{te}^2} - \left(\cancel{1} - \frac{(v-u)^2}{v_{te}^2}\right)\right]$$

$$= \frac{n_0}{\sqrt{\pi}\,v_{te}^3}\int_0^v dv\,\left(v^2 - 2vu + u^2 - \tfrac{1}{4}u^2\right)$$

$$= \frac{n_0}{\sqrt{\pi}\,v_{te}^3}\left[\frac{v^3}{3} - v^2 u + \frac{3}{4}u^2 v\right]$$

$$= \frac{n_0}{3\sqrt{\pi}\,v_{te}^3}\left(v - \frac{3u}{2}\right)^2 v$$

Note that this is zero at either end of the plateau.

$$\sum_k g(k)\,\delta(\omega_k - kv) = \frac{L}{2\pi}\int dk\, g(k)\,\delta(\omega_k - kv)$$

$$= \frac{L}{2\pi}\,\frac{g(k_*)}{\left|v - \frac{d\omega_k}{dk}\right|}\Bigg|_{v_p = v}$$

Easier to write $v = v_p$ rather than defining

$k(v)$ such that $\omega_{k_v} - k_v v = 0$

$$V_g = \frac{d\omega_k}{dk}$$

$$\frac{1}{m_e}\, k^2 \frac{|E_k|^2}{4\pi} \frac{\omega_{pi}^2}{\omega_0^3} \frac{L}{2\pi} \frac{1}{V_p - V_g} = \frac{n_0}{3\sqrt{\pi}\, V_{te}^3} V_p \left(V_p - \frac{3u}{2}\right)^2$$

$$\boxed{\frac{|E_k|^2}{4\pi} = \frac{n_0}{3\sqrt{\pi}} \frac{\Delta k}{k} \frac{\omega_0^2}{\omega_{pi}^2} \frac{V_p^2 (V_p - V_g)}{V_{te}^3}\, m_e \left(V_p - \frac{3u}{2}\right)^2}$$

where $\Delta k = \frac{2\pi}{L}$ $\quad V_p = \frac{C_s}{1 + k^2/k_{De}^2}$

Note that $|E_k|^2$ is zero at $V_p = 0$ $(k \to \infty)$

and $V_p = \frac{3u}{2}$ $\Rightarrow$ the ends of the plateau