The impact of frustrated singularities on magnetic island evolution

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The growth of magnetic islands is explored using the magnetohydrodynamic (MHD) model in a simple slab system in which the value of the tearing mode stability parameter Δ' can be varied continuously. Unless the system is close to marginal stability reconnection is controlled by Sweet-Parker current layers, whose formation is a consequence of the inherent singular structure of magnetic island equilibria.

The growth of magnetic islands in high temperature fusion plasmas can have a deleterious impact on energy confinement. The quasi-static island growth model of Rutherford [1] has been widely used to study the formation of such islands. In this model ion inertia is neglected and a simple equation for the time dependence of the island width w is obtained

$$dw/dt = (\eta c^2/4\pi)\Delta'. \tag{1}$$

The theory leading to this expression breaks down for the large values of the parameter Δ' governing the growth of the islands causing the sawtooth collapse in the core of tokamak plasmas[2, 3]. However, the conditions under which the Rutherford theory is valid have not been fully explored.

We evolve the resistive MHD equations in a slab current layer to study the growth of magnetic islands. In this system the tearing mode stability parameter Δ' can be varied in a controlled manner so that the validity of Rutherford theory can be tested. Current layers form near the magnetic x-line that develops during reconnection and increase in length and strength with increasing Δ'. Such current layers result from flux conservation during the release of magnetic energy which forces singularities in the magnetic island equilibria facilitated by reconnection. This behavior is related to that identified previously during reconnection leading to the sawtooth crash[2, 3]. The formation of the current layers causes Rutherford theory to break down even for rather small values of Δ'. A modified Sweet-Parker model is developed to describe magnetic island growth in this regime. More generally, we suggest that the dominance of Sweet-Parker reconnection rather than the open x-line configuration of Petchek[4] in resistive MHD simulations[5] is a consequence of the singular nature of the underlying ideal MHD reconnected state.

Simulations are completed with the full set of resistive MHD equations:

$$\frac{\partial n}{\partial t} + \nabla\cdot n\mathbf{v} = 0, \tag{2}$$

$$\frac{\partial (n\mathbf{v})}{\partial t} + \nabla\cdot(n\mathbf{v}\mathbf{v}) = \mathbf{J}\times\mathbf{B} - T\nabla n, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}, \tag{4}$$

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B} + \eta\mathbf{J}, \tag{5}$$

$$\mathbf{J} = \nabla\times\mathbf{B}, \tag{6}$$

where we have taken an isothermal approximation. The initial equilibrium is a periodic double current layer with a magnetic field given by $\mathbf{B} = \hat{z}\times\nabla\psi + \hat{z}B_z$ with $\psi = \sin(y)$ and B_z chosen so that $B^2 \equiv B_{z0}^2 = B_x^2 + B_z^2$ is constant. The density is therefore initially a constant value n_0. The equations are written in normalized units: magnetic fields to the peak value B_0 of B_x; space to $\lambda = L_y/2\pi$ with L_y the periodicity length in the y direction; and times to the Alfvén transit time λ/c_{A0} with $c_{A0} = B_0/\sqrt{4\pi m_i n_0}$. Simulations are carried out with $B_{z0} = 5.0$, which is in the large guide field limit, with $T = 1$, corresponding to $\beta = 2nT/B^2 = 0.08$. A small initial magnetic perturbation of $B_y \sim 10^{-5}$ is applied to initiate reconnection. In addition to the plasma resistivity, small fourth order viscous and diffusion terms are included to reduce fluctuations at the gridscale. For a computational domain of length L_x in the x direction the tearing mode stability parameter for the lowest order mode is given by $\Delta'\lambda = 2\kappa\tan(\pi\kappa/2)$, with $\kappa = \sqrt{1 - L_y^2/L_x^2}$[6]. The value of Δ' increases monotonically with increasing L_x from the marginal stability boundary $L_x = L_y$, which facilitates a study of magnetic island growth in both weakly and strongly unstable regimes.

In Fig. 1 we show the time dependence of the width of the magnetic island w for several values of Δ' (box length L_x) for $\eta = 2.8\times10^{-4}$. Consistent with linear theory [7] and Rutherford nonlinear theory, the islands grow faster at larger values of Δ'. However, in Rutherford theory the slopes of the curves in Fig. 1 should asymptote to constant values at late time, which is clearly not the case except for the smallest value of Δ'.

To understand the reasons for the deviation from Rutherford theory we show in Fig. 2 the out of plane current density, J_z, at comparable island widths from the simulations in Fig. 1. A distinct current sheet forms in

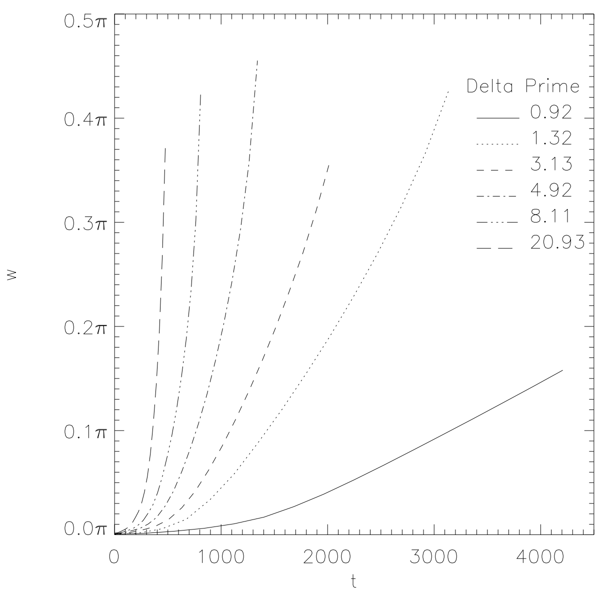

FIG. 1: Magnetic island half width w versus time for Δ' equal to 0.92, 1.32, 3.13, 4.92, 8.11, and 20.93.

the vicinity of the magnetic x-line for all but the smallest value of Δ'. The length of the current sheet and the magnitude of the J_z in the sheet and along the separatrix increase with Δ'. The formation of these current sheets is consistent with low resistivity reconnection studied earlier[8]. Further, the current sheets are reminiscent of the Sweet-Parker model of reconnection[9, 10], in which the rate of reconnection is limited by plasma outflow from the x-line (V_{out}), given by the Alfvén velocity based on the magnetic field just upstream from the current layer (c_{Aup}). Thus, the distinguishing feature of the Rutherford and the Sweet-Parker models is the presence or absense of a relation between V_{out} and c_{Aup}. In Fig. 3, V_{out} is plotted against c_{Aup} (or B_{up} since n is nearly constant) for various values of Δ'. B_{up} is the value of B_x taken upstream from the x-line a distance equal to twice the width of the current layer at half its maximum value. V_{out} is the maximum flow speed along a line between the x and o lines. Each line in Fig. 3 is a time sequence of the data points from a simulation with a given value of Δ'. In each case time runs upward toward higher velocity. For the larger values of Δ' a clear linear relation between V_{out} and B_{up} is established as the velocity increases. This indicates that for these cases the outflow velocity from the x-line is being limited by the Alfvén velocity as in the Sweet-Parker theory. For the two smallest values of Δ', 0.92 and 1.32, V_{out} remains small until the island impinges on the neighboring x line. The current layers in these cases are not sufficiently well developed for the outflows to be limited by the Alfvén speed.

To further test the consistency of larger Δ' island growth with the Sweet-Parker model, we have varied the value of η by a factor of four and measured the impact on the reconnection electric field, E_r. For the two largest values of Δ', 8.11 and 20.93, E_r scales like $\eta^{0.53}$ and $\eta^{0.51}$, respectively, which is close to the Sweet-Parker scaling, $E_r \propto \eta^{1/2}$.

The formation of the current layers in Fig. 2 is also reminiscent of the equilibrium theory of Waelbroeck, who showed that in the case of $m = 1$ reconnection in tokamaks, which corresponds to a very large value of Δ', the conservation of magnetic flux implies the formation of a current layer[2, 3]. To test whether the current layers in Fig. 2 have a similar origin, for several values of Δ' we set η and the plasma flows to zero during the island growth phase, thereby halting reconnection. The island equilibrium was allowed to relax with viscosity included to damp flows. In all cases the current layers sharpened in a secular manner, preserving their integrated value, with only small changes in length. Shown in Fig. 4 is J_z late in the equilibrium calculation for $\Delta' = 20.93$ for three island widths (note the difference in scale in the y direction). The peak current at the x-line in the largest island is double that for the same value of Δ' in Fig. 1 and continues to increase until the layer can no longer be resolved with our grid. The reference to these current layers as "frustrated singularities" is intended to differentiate the behavior seen in our simulations, which because of resistivity is not literally singular, from the unbounded behavior resulting from a completely ideal process. Waelbroeck's theory for the $m = 1$ mode predicts that the magnetic island evolves in a self-similar manner so that the length of the current layer is independent of the island width. Such self-similar behavior is evident in the islands shown in Fig. 4.

A fundamental question is why current layers form during the growth of magnetic islands and what controls their length? We suggest that the length and integrated current of the layer, are a consequence of area and flux preservation, as explored by Waelbroeck for the $m = 1$ case, combined with the requirement that reconnection reduce the magnetic energy in the system and that these concepts are applicable to the growth of magnetic islands shown in Fig. 2. We illustrate this basic idea with the very simple model of magnetic reconnection shown in Fig. 5. In Fig. 5 a reversed magnetic field $B_i(y) = B_i' y$ in a system of length L reconnects flux $B_i' y^2/2$. The reconnected flux forms a magnetic island, represented by a field $B_w = B_w' w$ distributed over a width w and length L_w. The length of the current layer L_J is therefore $L - L_w$. The idea is that the only way for magnetic energy to be released while preserving the total flux is for the island to expand radially. This causes the field strength to decrease. Incompressibility, however, at the same time requires L_w to decrease below L, forcing a current layer to develop. Applying these arguments to Fig. 5, we find

$$L_J = L(1 - \sqrt{U_f/U_i}), \qquad (7)$$

where U_i and U_f are the initial and final energies, respectively.

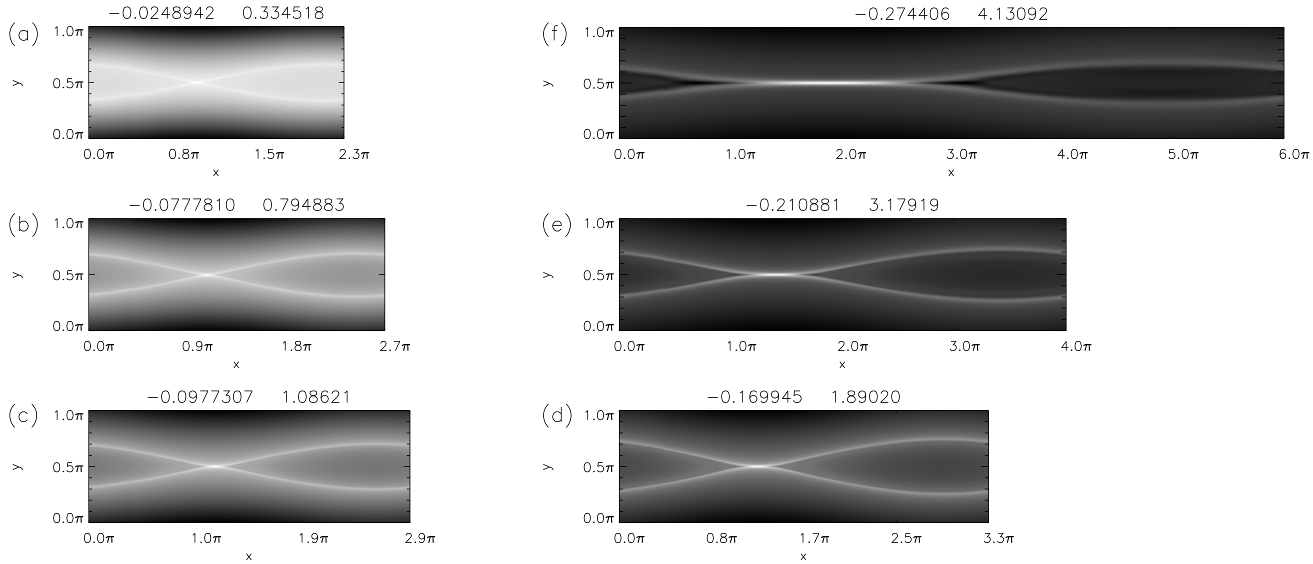

FIG. 2: J_z in the $x-y$ plane for Δ' equal to (a) 0.92, (b) 1.32, (c) 3.13, (d) 4.92, (e) 8.11, (f) 20.93

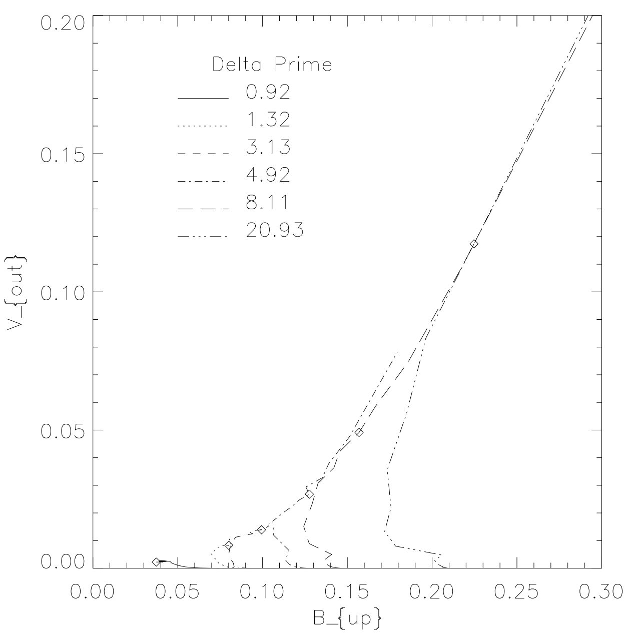

FIG. 3: Time sequence of the maximum outflow velocity, V_{out}, versus the upstream magnetic field, B_{up}, for several values of Δ'. Time increases upwards on each line. The diamond on each curve marks the upstream field and outflow at an island width $w = 0.16\pi$.

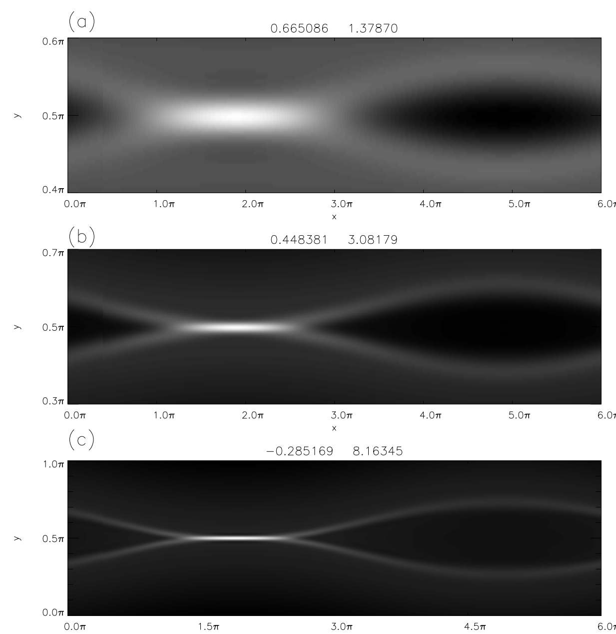

FIG. 4: Relaxed state current distribution J_z for $\Delta' = 20.93$ for $(a) w = 0.04\pi$,$(b) w = 0.1\pi$ and $(c) w = 0.22\pi$.

To understand the rate of island growth in simulations where the outflow rates from the current layers are limited to the Alfvén speed we generalize the Sweet-Parker model for the specific geometry of these current layers. Continuity for nearly incompressible flow yields

$$v_{in} = (\delta/L_J)c_{Aup}. \quad (8)$$

where $\delta = \eta/v_{in}$ is the width of the resistive layer at the x-line[9, 10]. At first glance Eq. (8) would seem to contradict the results of simulations shown in Fig. 1 since v_{in} would be larger for the smaller value of L_J at small Δ' (see Fig. 2). However, the reduction of L_J with small Δ' can be offset by a corresponding reduction of c_{Aup}. That this is the case can be seen in Fig. 3 where the diamond marks the value of B_{up} at an island width of $w = 0.16\pi$ for each value of Δ'. The value of B_{up} decreases with decreasing Δ'. In the Waelbroeck equilibrium theory the

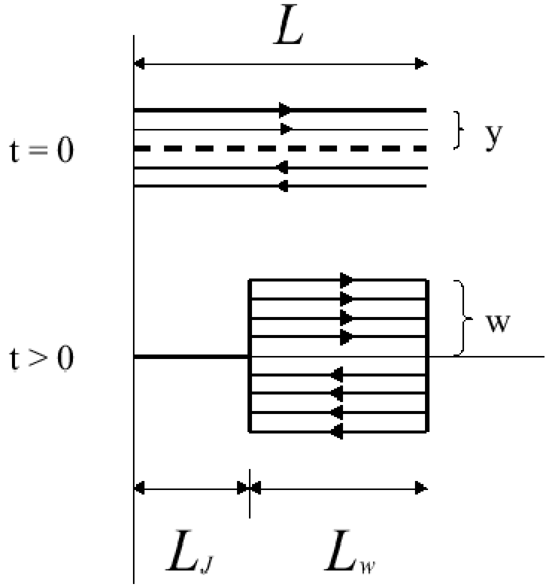

FIG. 5: Model of magnetic reconnection and the formation of the Sweet-Parker current layer.

integrated current across the separatrix at the center of the island is related to that across the x-line, implying that the magnetic field upstream of the x-line is given by $\Delta B = B_i - B_w$. For notational simplicity we take $\Delta B \ll B_i$ although aside from factors of order unity the final expression for island growth is valid for arbitrary ΔB. The length of the current layer L_J can be expressed in terms of this jump in magnetic field as

$$L_J = L\Delta B/B_i. \tag{9}$$

Inserting $c_{Aup} = (\Delta B/B_i)c_{Ai} = (L_J/L)c_{Ai}$ into Eq. (8), we obtain the expression

$$v_{in} = (\delta/L)c_{Ai}, \tag{10}$$

where c_{Ai} is now the Alfvén speed upstream from the magnetic island and L is the macroscopic system length. The dependence on ΔB has dropped out of this expression. However, the reconnection electric field, $v_{in}\Delta B/c$, depends explicitly on the upstream magnetic field ΔB and therefore on Δ'. The reconnection electric field is equal to the rate of increase of magnetic flux in the island $B_w\dot{w} \simeq B_i\dot{w}$ so

$$\dot{w} = v_{in}\Delta B/B_i. \tag{11}$$

For small magnetic islands we can relate ΔB to the tearing mode stability parameter Δ'. The flux perturbation is

$$\tilde{\psi} = \tilde{\psi}_0 \cos(kx)(1 + \Delta' y/2), \tag{12}$$

where $\tilde{\psi}_0$ is linked to the island width

$$\tilde{\psi}_0 = -\frac{B_i' w^2}{4}\frac{1}{1 + \Delta' w/4}. \tag{13}$$

The magnetic field upstream of the x-line is then given by

$$\Delta B = -\tilde{\psi}_0\frac{\Delta'}{2} = \frac{B_i' w}{2}\frac{\Delta' w/4}{1 + \Delta' w/4}. \tag{14}$$

This expression for ΔB combined with Eq.(9) for L_J implies that $L_J \sim \Delta'$ for small Δ', which is consistent with the simulations. The final equation for the growth of small islands is given by

$$\dot{w} = \frac{L}{(\tau_r\tau_{Ai})^{1/2}}\frac{\Delta' w/4}{1 + \Delta' w/4}, \tag{15}$$

where $\tau_r = 4\pi L^2/\eta c^2$ is the resistive time and the Alfvén time $\tau_{Ai} = L/c_{Ai}$ is evaluated with the magnetic field $B_i \sim w$. For $\Delta' w/4 > 1$, this equation reduces to the Sweet-Parker scaling for reconnection. The results are a generalization of the Sweet-Parker model to systems in which the tearing mode stability parameter Δ' is not large.

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