Math 125 Practice Final Exam: Fall 1999

1. [25pts] Compute the following derivatives.

a) $\frac{d}{dx}5^{\sin(x)} = \ln(5)5^{\sin(x)}\cos(x)$

b) $\frac{d}{dt}\left(\frac{t+t\ln(t)}{\cosh(t)}\right) = \frac{\cosh(t)(1+\ln(t)+1)-(t+t\ln(t))\sinh(t)}{\cosh^2(t)}$ (by the quotient rule)

c) $\frac{d}{dq}\sqrt{1+e^{\tan(q)}} = \frac{1}{2}\left(1+e^{\tan(q)}\right)^{-\frac{1}{2}} e^{\tan(q)}\sec^2(q)$

d) $\frac{d}{ds}\arctan(s^3+1) = \frac{1}{1+(s^3+1)^2}\,3s^2.$

e) $\frac{d}{dz}\left(z\cos(2z)+\sin(z^2)\right) = (\cos(2z)+z(-\sin(2z)2)) + \cos(z^2)2z$

2. [24pts] Compute the following indefinite integrals. Indicate any substitutions or identities that you use to bring them into elementary forms.

a) $\int \sin(3t+5)\,dt = -\frac{1}{3}\cos(3t+5)+C$

By the substitution $u=3t+5,\ du=3\,dt,$ the integral has the form $\frac{1}{3}\int \sin(u)\,du = -\frac{1}{3}\cos(u)+C.$

b) $\int e^x\cosh(e^x)\,dx = \sinh(e^x)+C$

By the substitution $u=e^x,\ du=e^x\,dx,$ the integral has the form $\int \cosh(u)\,du = \sinh(u)+C.$

c) $\int \frac{\cos(z^{\frac{1}{3}})}{z^{\frac{2}{3}}}\,dz = 3\sin(z^{\frac{1}{3}})+C$

By the substitution $u=z^{\frac{1}{3}},\ du=\frac{1}{3}z^{-\frac{2}{3}}\,dz,$ the integral has the form $3\int \cos(u)\,du = 3\sin(u)+C.$

d) $\int \frac{v}{1+v^4}\,dv = \frac{1}{2}\arctan(v^2)+C$

By the substitution $u=v^2,\ du=2v\,dv,$ the integral has the form $\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u)+C.$

3. [10pts] Find the equation of the tangent line to the curve $x^3+y^2\cos(y-1)=2$ at the point (1,1).

Solution: Differentiate the equation for the curve implicitly to find

$$3x^2+\left(2y\cos(y-1)-y^2\sin(y-1)\right)\frac{dy}{dx}=0\,.$$

Evaluate this at the point $(x,y)=(1,1)$ to obtain

$$3+2\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}=0\,.$$

Thus the slope of the tangent line at (1,1) is $-\frac{3}{2}$ and the equation of the tangent line is

$$y-1=-\frac{3}{2}(x-1) \qquad \text{or} \qquad y=1-\frac{3}{2}(x-1)\,.$$

4. [5pts] Compute $\dfrac{d}{dr}\displaystyle\int_0^{r^2} \sec(t)\,dt$.

Solution: Use the second fundamental theorem of calculus and the chain rule to obtain

$$\frac{d}{dr}\int_0^{r^2} \sec(t)\,dt = \sec(r^2)\,2r\,.$$

5. [10pts] Suppose the function f is continuous over $[a, b]$ and that its derivative is has the graph depicted below. Identify the intervals over which f is increasing, decreasing, concave up, and concave down.

Solution: The graph of f' is not reproduced here, but from it one reads off that f is:

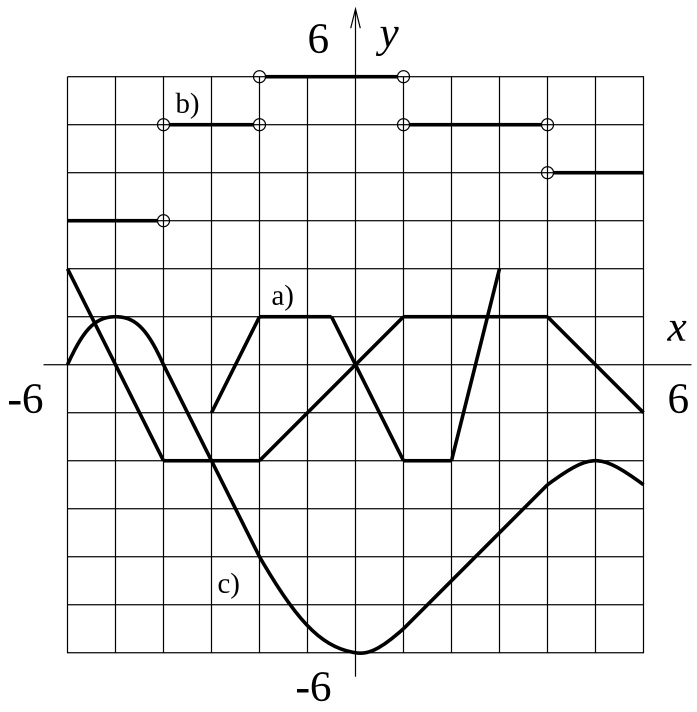
i) increasing over $[a, x_2]$ and $[x_3, b]$ (because f' is positive inside each);

ii) decreasing over $[x_2, x_3]$ (because f' is negative inside);

iii) concave up over $[a, x_1]$ and $[x_4, x_6]$ (because f' is increasing inside each);

iv) concave down over $[x_1, x_3]$, $[x_3, x_4]$, and $[x_6, b]$ (because f' is decreasing inside each).

6. [20pts] Consider the function defined over [-6,6] whose graph is shown below.



Using the same axes do the following:

a) Sketch $y = f(-2x)$;

b) Sketch $y = f'(x) + 5$;

c) Sketch $y = F(x)$ where $F'(x) = f(x)$ and $F(0) = -6$.

d) For what values of x does the function F in part (c) have either a local or global maximum or mimimum. Indicate your reasoning.

Solution: The answers for parts a-c are shown on the graph above. The answer for part d can either be surmised from the answer to part c or read directly from the given graph of f. The local extrema of F are:

i) a local min at $x = -6$ (left endpoint with positive slope),

ii) a local max at $x = -5$ (positive to negative slope),

iii) a local min at $x = 0$ (negative to positive slope),

iv) a local max at $x = 5$ (positive to negative slope),

v) a local min at $x = 6$ (right endpoint with negative slope).

Of these, the global extrema of F are:

i) a global max at $x = -5$ (negative total change from -5 to 5),

ii) a global min at $x = 0$ (negative total change from -6 to 0 and positive from 0 to 6).

7. [20pts] A box with a square base and no top is to be constructed with a volume of 3 cubic meters. If the material for the bottom costs \$1.60 per square meter, and the material for the sides costs \$.90 per square meter, what are the dimensions of the box that minimize the total cost of the material needed for its construction?

Solution: Let x denote the length of the edges of the base of the box and let h denote the height of the box. Because the area of the fours sides is $4xh$ while the area of the bottom is x^2, the cost C of the materials for the box is

$$C = .9 \cdot (4xh) + 1.6 \cdot x^2 \,.$$

To minimize this cost we first eliminate h by using the fact that the volume of the box, which is x^2h, was specified in the problem to be 3 cubic meters. This gives

$$x^2h = 3 \,, \qquad \text{or} \qquad h = \frac{3}{x^2} \,,$$

which implies that the cost as a function of x is

$$C = .9 \cdot 4x\frac{3}{x^2} + 1.6x^2 = 10.8x^{-1} + 1.6x^2 \,.$$

This function is to be minimized over the interval $(0, +\infty)$. Notice that it tends to ∞ as $x \to 0^+$ and as $x \to +\infty$. It also has a single critical point (which must therefore be a minimum) where

$$\frac{dC}{dx} = -10.8x^{-2} + 3.2x = 0 \,, \qquad \text{or} \qquad x = \left(\frac{10.8}{3.2}\right)^{\frac{1}{3}} = \left(\frac{27}{8}\right)^{\frac{1}{3}} = \frac{3}{2} \,.$$

At this point

$$h = \frac{3}{x^2} = 3\left(\frac{2}{3}\right)^2 = \frac{4}{3} \,.$$

Hence, the dimensions of the box that minimize the total cost of its construction are

a square base of $\frac{3}{2}$ meters by $\frac{3}{2}$ meters and a height of $\frac{4}{3}$ meters.

8. [18pts] Evaluate the following definite integrals exactly. Indicate how your answer was obtained.

a) $\displaystyle\int_0^2 \frac{e^x}{3+e^x}\,dx = \ln(3+e^2) - \ln(4) = \ln\left(\frac{3+e^2}{4}\right)$.

By the substitution $u = 3+e^x$, $du = e^x\,dx$, the intgral has the form $\displaystyle\int \frac{1}{u}\,du = \ln(|u|) + C$. Then either do

$$\int_0^2 \frac{e^x}{3+e^x}\,dx = \ln(3+e^x)\Big|_0^2 = \ln(3+e^2) - \ln(4)\,,$$

or use the change of variables formula to do

$$\int_0^2 \frac{e^x}{3+e^x}\,dx = \int_4^{3+e^2} \frac{1}{u}\,du = \ln(u)\Big|_4^{3+e^2} = \ln(3+e^2) - \ln(4)\,.$$

b) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-q^2}}\,dq = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

The integral already has the elementary form $\displaystyle\int \frac{1}{\sqrt{1-q^2}}\,dq = \arcsin(q) + C$.

So just do

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-q^2}}\,dq = \arcsin(q)\Big|_0^{\frac{1}{2}} = \arcsin\left(\frac{1}{2}\right)\,.$$

c) $\displaystyle\int_{-\pi}^{\pi} \left(t^2 + \sin^3(t^2)\,t\right)dt = \frac{2\pi^3}{3}$.

The easy way to do this is through symmetry. First observe that the interval of integration $[-\pi,\pi]$ is symmetric. Then observe that the intgrand is the sum of an even function t^2, whose contribution is

$$\int_{-\pi}^{\pi} t^2\,dt = 2\int_0^{\pi} t^2\,dt = \frac{2}{3}t^3\Big|_0^{\pi} = \frac{2}{3}\pi^3\,,$$

and an odd function $\sin^3(t^2)t$ whose contribution is zero.

A harder way to do this (if you did not see the symmetry) is to break it into two parts. The contribution of the first is

$$\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{3}t^3\Big|_{-\pi}^{\pi} = \frac{1}{3}\pi^3 - \frac{1}{3}(-\pi)^3 = \frac{2}{3}\pi^3\,.$$

The contribution of the second can be obtained in a number of ways. By the substitution $u = \cos(t^2)$, $du = \sin(t^2)2t\,dt$, and the identity $\sin^2(t^2) = 1 - \cos^2(t^2) = 1 - u^2$, the integral has the form $\displaystyle\frac{1}{2}\int (1-u^2)\,du = \frac{1}{2}u - \frac{1}{6}u^3 + C$. Then either do

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^3(t^2)\,t\,dt &= \left(\frac{1}{2}\cos(t^2) - \frac{1}{6}\cos^3(t^2)\right)\Big|_{-\pi}^{\pi} \\ &= \left(\frac{1}{2}\cos(\pi^2) - \frac{1}{6}\cos^3(\pi^2)\right) - \left(\frac{1}{2}\cos(\pi^2) - \frac{1}{6}\cos^3(\pi^2)\right) = 0\,,\end{aligned}$$

or use the change of variables formula to do

$$\int_{-\pi}^{\pi} \sin^3(t^2)\,t\,dt = \frac{1}{2}\int_{\cos(\pi^2)}^{\cos(\pi^2)} (1-u^2)\,du = 0\,.$$

9. [8pts] Consider the region $\mathcal{R}$ bounded by $x = -1$ and $x = +1$ and the curves $y = \cosh(x)$ and $y = \sinh(x)$.

a) Find the area of the region $\mathcal{R}$.

Solution: A slice of $\mathcal{R}$ at location x (on the x-axis) with thickness Δx has an area approximately given by $(\cosh(x) - \sinh(x))\Delta x$. By passing to the limit $\Delta x \to 0$ in the Riemann sums, one arrives at the integral

$$\int_{-1}^{1} (\cosh(x) - \sinh(x))\, dx = (\sinh(x) - \cosh(x))\Big|_{-1}^{1} = 2\sinh(1)\,.$$

b) *Set up* a definite integral for the volume of the solid with base $\mathcal{R}$ and cross sections perpendicular to the x-axis given by semi-circles. (You do not need to evaluate this integral).

Solution: At location x on the x-axis a slice of this solid has cross sectional area equal to half the area of a circle with diameter equal to $(\cosh(x) - \sinh(x))$. Thus a slice of thickness Δx has a volume approximately given by

$$\frac{1}{2}\pi \left(\frac{\cosh(x) - \sinh(x)}{2}\right)^2 \Delta x = \frac{\pi}{8}(\cosh(x) - \sinh(x))^2 \Delta x\,.$$

By passing to the limit $\Delta x \to 0$ in the Riemann sums, one arrives at the integral

$$\frac{\pi}{8}\int_{-1}^{1} (\cosh(x) - \sinh(x))^2\, dx\,.$$

We remark that (even though it is not asked for) this integral is easy to evaluate if you notice the identity $\cosh(x) - \sinh(x) = e^{-x}$, in which case the integral becomes

$$\frac{\pi}{8}\int_{-1}^{1} e^{-2x}\, dx = -\frac{\pi}{16}e^{-2x}\Big|_{-1}^{1} = \frac{\pi}{16}(e^2 - e^{-2}) = \frac{\pi}{8}\sinh(2)\,.$$

10. [18pts] A toy car is propelled by a wind-up motor. At time $t = 0$ it is released with zero velocity. Its acceleration in meters/sec^2 as a function of time for $t \geq 0$ is given by

$$a(t) = \begin{cases} 2 - t & \text{for } 0 \leq t \leq 4\,, \\ 0 & \text{for } t > 4\,. \end{cases}$$

a) Find $v(t)$, the velocity of the car as a function of time, for $t \geq 0$.

Solution: For $0 \leq t \leq 4$ we have

$$v(t) = \int_0^t a(s)\, ds = \int_0^t (2 - s)\, ds = 2t - \frac{1}{2}t^2\,.$$

Because the acceleration is zero for $t > 4$, the velocity stays equal to $v(4) = 0$ for $t > 4$. Hence, in meters/sec

$$v(t) = \begin{cases} 2t - \frac{1}{2}t^2 & \text{for } 0 \leq t \leq 4\,, \\ 0 & \text{for } t > 4\,. \end{cases}$$

b) Find $s(t)$, the distance the car travels as a function of time, for $t \geq 0$.

Solution: For $0 \leq t \leq 4$ we have

$$s(t) = \int_0^t v(s)\, ds = \int_0^t \left(2s - \frac{1}{2}s^2\right) ds = t^2 - \frac{1}{6}t^3 .$$

Because the velocity is zero for $t > 4$, the distance traveled stays equal to $s(4) = 16/3$ for $t > 4$. Hence, in meters

$$s(t) = \begin{cases} t^2 - \frac{1}{6}t^3 & \text{for } 0 \leq t \leq 4 , \\ \frac{16}{3} & \text{for } t > 4 . \end{cases}$$

11. [6pts] Suppose that you are using Newton's method to find the roots of the function shown below. To which root, if any, will the method converge if your starting value is x_0? Give reasons for your answer.

Solution: (The graph of f is not reproduced here, so please refer to the one on the Practice Exam.) Newton's method will converge to the root a on the left. The reason is that the tangent line at x_0 intersects the x-axis to the left of a. The concavity of the graph suggests that the subsequent tangents all stay to the left of the curve and that the approximation will converge to a from below.

12. [10pts] Consider the family of functions $f(t) = te^{-at}$ defined for $t \geq 0$. For what value of a does the maximum value of f occur at $t = 2$?

Solution: Each member of this family is differentiable over $[0, \infty)$, satisfies $f(0) = 0$, and is positive over $(0, \infty)$. If $a \leq 0$ then $f(t) \to \infty$ as $t \to \infty$, so that no maximum value exists. If $a > 0$ then $f(t) \to 0$ as $t \to \infty$, so that f will take on a maximum value over $(0, \infty)$. In that case the maximum will occur at point where

$$f'(t) = \frac{d}{dt}\left(te^{-at}\right) = e^{-at}(1 - at) = 0 .$$

This only happens at $t = 1/a$. To make this point equal to 2 one must choose $a = 1/2$.

13. [18pts] Consider the definite integral $\displaystyle\int_0^{\frac{4}{5}} e^{-\frac{1}{2}x^2}\, dx$.

a) Use ALLSUMS to approximate this integral by the left-hand, right-hand, trapezoid, and midpoint rules with 4 and 128 uniform subintervals. Give Δx and complete the following table to 6 decimal accuracy.

subintervals	4	128
Δx	$\frac{1}{5}$	$\frac{1}{160}$
LEFT	.747717	.723125
RIGHT	.692947	.721414
TRAP	.720332	.722270
MID	.723242	.722272

b) In the case of 4 uniform subintervals, what is the third subinterval and the exact height of the corresponding rectangle for the left-hand rule?

Solution: The third subinterval is $[\frac{2}{5}, \frac{3}{5}]$ and the exact height of the corresponding rectangle for the left-hand rule is $e^{-\frac{1}{2}(\frac{2}{5})^2}$.

c) Do the trapezoid and midpoint rules give either over or under estimates for this integral? Why?

Solution: The trapezoid rule gives an underestimate and the midpoint rule an overestimate because the integrand f is *concave down* over $[0, \frac{4}{5}]$. The concavity of f can either be argued from its graph or be argued directly from $f'(x) = -xe^{-\frac{1}{2}x^2}$, so that $f''(x) = -(1-x^2)e^{-\frac{1}{2}x^2} < 0$ over $[0, \frac{4}{5}]$.

d) How many subintervals are needed to get 5 decimal place accuracy for the right-hand rule?

Solution: To get 5 decimal place accuracy one must insure that the error of the right-hand rule is less than .000005 ($= 1/200000$). Because the integrand f is *decreasing* over $[0, \frac{4}{5}]$, the error of the right-hand rule is less than $|f(\frac{4}{5}) - f(0)|\Delta x$ where $\Delta x = \frac{4}{5}/n$ for n subintervals. So if one picks n such that

$$(f(0) - f(\tfrac{4}{5}))\Delta x = \left(1 - e^{-\frac{1}{2}(\frac{4}{5})^2}\right)\frac{4}{5n} < .000005 = \frac{1}{200000},$$

then the error of the right-hand rule will be less than .000005. But this is equivalent to

$$n > \left(1 - e^{-\frac{1}{2}(\frac{4}{5})^2}\right)\frac{4}{5}200000 = \left(1 - e^{-\frac{1}{2}(\frac{4}{5})^2}\right)160000 > 43816.154\,.$$

Any number between 43817 and 160000 would be an acceptable answer depending on how you approximated in the last step.