SOME APPLICATIONS OF DERIVATIVES

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20 October 1999

This is a review of some of the material we have covered pertaining to the application of derivatives and which will be covered on the third exam. It supplements the material covered in the book (Chapter 5 and Appendices A and E) and in the class lectures. It includes minima and maxima, monotonicity and concavity, inflections, general optimization problems, economic optimization problems, tangent line approximations, root finding, and the Newton-Raphson method.

1. FINDING MINIMA AND MAXIMA

1.1: Minima and Maxima. Consider a function f that is defined over an interval I, where I is either (a,b), $[a,b)$, $(a,b]$ or $[a,b]$ for some $a < b$. Here we allow $a = -\infty$ or $b = \infty$ as possible open endpoints. We say that f has a **minimum** over I if there exists a point p in I such that

$$f(p) \leq f(x) \quad \text{for every } x \text{ in } I\,.$$

While there can be more than one such p, it is clear that the value $f(p)$ must be unique. Given such a p, we say

- f has a **minimum** over I at p,
- $f(p)$ is the **minimum value** of f over I.

Similarly, we say that f has a **maximum** over I if there exists a p in I such that

$$f(x) \leq f(p) \quad \text{for every } x \text{ in } I\,.$$

While there can be more than one such p, it is again clear that the value $f(p)$ must be unique. Given such a p, we say

- f has a **maximum** over I at p,
- $f(p)$ is the **maximum value** of f over I.

Points that are either a minimum or a maximum of f over I are called **extreme points** or **extrema** of f over I and their corresponding values are called **extreme values** of f over I.

A general function f defined over an interval I may have neither a minimum nor a maximum. For example, consider

$$\begin{aligned}
f(x) &= \tanh(x) && \text{over } (-\infty,\infty)\,,\\
f(x) &= \tan(x) && \text{over } (-\tfrac{\pi}{2},\tfrac{\pi}{2})\,,\\
f(x) &= x^3 && \text{over } (-\infty,\infty)\,.
\end{aligned}$$

Some may have one but not the other. For example, consider

$$\begin{aligned}
f(x) &= \operatorname{sech}(x) && \text{over } (-\infty,\infty)\,,\\
f(x) &= \sec(x) && \text{over } (-\tfrac{\pi}{2},\tfrac{\pi}{2})\,,\\
f(x) &= x^2 && \text{over } (-\infty,\infty)\,.
\end{aligned}$$

And some may have both. For example, consider

$$\begin{aligned} f(x) &= \sin(x) && \text{over } (-\infty, \infty)\,, \\ f(x) &= \frac{x}{1+x^2} && \text{over } (-\infty, \infty)\,, \\ f(x) &= xe^{-x} && \text{over } [0, \infty)\,. \end{aligned}$$

There is one general theorem that you should know.

Min-Max Theorem: If f is a *continuous* function defined over a *closed* interval $[a, b]$ then there exists points p_{min} and p_{max} in $[a, b]$ such that

$$f(p_{min}) \leq f(x) \leq f(p_{max}) \quad \text{for every } x \text{ in } [a, b]\,.$$

In other words, f has both a minimum and a maximum over $[a, b]$.

This fact we will take as a given; its proof is beyond the scope of this course. It is not true in general if $[a, b]$ is replaced by either $(a, b]$, $[a, b)$ or (a, b), or if f is not continuous over $[a, b]$. Can you illustrate this with examples?

1.2: Local Minima and Maxima. Let f be defined over an interval I, where I is either (a, b), $[a, b)$, $(a, b]$ or $[a, b]$ for some $a < b$. If we are going to hunt for a minimum or maximum of f over I, it is helpful to narrow our search. We say that f has a **local minimum** at a point p in I if

$$f(p) \leq f(x) \quad \text{over the intersection of } I \text{ with some small open interval containing } p\,.$$

Similarly, we say that f has a **local maximum** at a point p in I if

$$f(x) \leq f(p) \quad \text{over the intersection of } I \text{ with some small open interval containing } p\,.$$

It is clear that a minimum or a maximum of f over I, if it exists, must be one of the local minima or local maxima respectively. In this context, a minimum or a maximum of f over I is referred to as a **global minimum** or a **global maximum** respectively. Points that are either a local minimum or a local maximum of f over I are called **local extreme points** or **local extrema** and their corresponding values are called **local extreme values**. One similarly defines **global extreme points**, **global extrema**, and **global extreme values**. Sometimes the terms **relative** and **absolute** are used in place of **local** and **global** respectively; they mean the same thing.

Local Extrema may be found with the help of the following very important fact.

Zero-Slope Theorem: Let f be defined over an interval (a, b). If

- p is a local extreme point of f over (a, b),
- f is differentiable at p,

then $f'(p) = 0$.

Reason: Because p is a local extreme point of f over (a, b), $f(p)$ is either a local minimum or a local maximum value of f over (a, b). Suppose $f(p)$ is a local minimum value. Then we have

$$\frac{f(p+h) - f(p)}{h} \geq 0 \quad \text{for } h > 0\,, \qquad \frac{f(p+h) - f(p)}{h} \leq 0 \quad \text{for } h < 0\,.$$

Because f is differentiable at p, this implies that

$$f'(p) = \lim_{h \to 0^+} \frac{f(p+h) - f(p)}{h} \geq 0\,, \qquad f'(p) = \lim_{h \to 0^-} \frac{f(p+h) - f(p)}{h} \leq 0\,,$$

whereby $f'(p) = 0$. The argument when $f(p)$ is a local maximum value goes similarly. //

Notice that the Zero-Slope Theorem is false if we replace the open interval (a, b) by either $(a, b]$, $[a, b)$ or $[a, b]$. For example, consider the function

$$f(x) = x \qquad \text{over } [-1, 1]\,.$$

It has a local minimum at -1 and a local maximum at -1, yet has $f'(x) = 1$ everywhere. The argument given above that $f'(p) = 0$ breaks down at endpoints because it uses both left and right sided limits.

To hunt for the extrema of a function f defined over an interval I, the Zero-Slope Theorem suggests that we can restrict our search to a study of the behavior of f at the endpoints of I and to those points p in (a, b) where either $f'(p) = 0$ or f' is undefined. If p is a point in I where either $f'(p) = 0$ or f' is undefined then

- p is called a **critical point** of f over I;
- $f(p)$ is called a **critical value** of f over I.

Sometimes the point $(p, f(p))$ on the graph of f is also called a critical point. The intended meaning should be drawn from the context. We will often work with differentiable functions, in which case our critical points will be of the $f'(p) = 0$ type.

1.3: Finding Extrema by Comparing Values. If f is a continuous function defined over a closed interval $[a, b]$ then the Min-Max Theorem states that f has both a global minimum and a global maximum over $[a, b]$. If f is differentiable over (a, b) then we can find these as follows:

- find the critical points of f by solving for $f'(p) = 0$;
- evaluate the corresponding critical values $f(p)$;
- evaluate the endpoint values, $f(a)$ and $f(b)$;
- the lowest of the critical and endpoint values is the global minimum value;
- the highest of the critical and endpoint values is the global maximum value.

Fortunately, this is a case that often arises in applications.

On the other hand, consider f to be a continuous function defined over an interval I that is either (a, b), $[a, b)$, or $(a, b]$. Recall that we allow $a = -\infty$ or $b = \infty$ as possible open endpoints. The Min-Max Theorem no longer applies, so great care must be taken. If f is differentiable over I and either a global minimum or global maximum exists then we can find it as follows:

- find the critical points of f by solving for $f'(p) = 0$;
- evaluate the corresponding critical values $f(p)$;
- evaluate any closed endpoint values, $f(a)$ or $f(b)$;
- determine the limiting behavior at any open endpoint;
- the lowest of the critical and endpoint values is the global minimum value, if no lower values are attained by the limiting behavior at an open endpoint;
- the highest of the critical and endpoint values is the global maximum value, if no higher values are attained by the limiting behavior at an open endpoint;

The existence of the global minimum and global maximum values depends crucially on the behavior of f near the open endpoints of I.

In either case these techniques also allow you to identify local extrema by applying them to f restricted to appropriate subintervals of I.

Example: Consider $f(x) = x^4 - 4x^3 - 2x^2 + 12x$ over $(-\infty, \infty)$. One has

$$f'(x) = 4x^3 - 12x^2 - 4x + 12 = 4(x-1)(x+1)(x-3) = 0\,.$$

The critical points are therefore -1, 1, and 3. The corresponding critical values are then -9, 7, and -9. The limiting behavior as $x \to \pm\infty$ is $f(x) \to +\infty$. Hence, there are global minima at $x = -1$ and $x = 3$, a local maximum at $x = 1$, but there is no global maximum.

2. MONOTONICITY, CONCAVITY, EXTREMA AND INFLECTIONS

2.1: Monotonicity, Critical Points and Extrema. Let I be either (a,b), $[a,b)$, $(a,b]$ or $[a,b]$ for some $a<b$. Recall that if a function f is defined over I then we say f is **increasing** over I if for every x and y in I

$$x<y \quad \text{implies} \quad f(x)<f(y)\,, \tag{2.1}$$

and we say f is **decreasing** over I if for every x and y in I

$$x<y \quad \text{implies} \quad f(x)>f(y)\,. \tag{2.2}$$

If f is either increasing or decreasing over I, it is said to be **monotonic** over I. Accordingly, we say the **monotonicity** of f over I is either increasing or decreasing. Of course, most functions are not monotonic over their whole domain. To "determine the monotonicity of a function" means to identify the intervals over which the function is increasing or decreasing. You should be able to do this for a function given by a formula by studying the sign of its first derivative. For example, we have the following theorem.

Monotonicity Theorem: If f is continuous over I and differentiable over (a,b), you can read off the following information about f from its first derivative:

- if $f'>0$ over (a,b) then f is increasing over I;
- if $f'<0$ over (a,b) then f is decreasing over I;
- if $f'=0$ over (a,b) then f is constant over I.

This theorem seems obvious, but its justification is not as easy as you might guess. We will just assume it is true here.

The Monotonicity Theorem says nothing about points where f' either vanishes or is undefined. Recall that such points are the critical points of f over I. It is a fact that f cannot change its monotonicity between successive critical points. Given a critical point p in the domain of f, we shall say that f is either increasing or decreasing **near** p if it is so over the intersection of the domain with some small open interval containing p.

A critical point p of a function f is called **isolated** if it is the only critical point of f in some small open interval containing p. You should be able to give examples of critical points that are isolated and of ones that are not isolated. For example, with the aid of your calculator, consider the function

$$f(x)=\begin{cases} x^2\cos\left(\dfrac{1}{x}\right) & \text{for } x\neq 0\,,\\ 0 & \text{for } x=0\,. \end{cases}$$

This function is differentiable over $(-\infty, \infty)$. It has many critical points p, each of which satisfy $f'(p) = 0$. All of them are isolated except the one at 0. Do you see that $f'(0) = 0$?

At an isolated critical point p exactly one of only four things can happen; that thing is determined by the **First Derivative Sign Test for Local Extrema:**

- if f' is negative on the left and positive on the right near p
  then f has a local minimum at p;
- if f' is positive on the left and negative on the right near p
  then f has a local maximum at p;
- if f' is positive on both the left and right near p
  then f is increasing near p;
- if f' is negative on both the left and right near p
  then f is decreasing near p.

Examples that illustrate each of these possibilities are respectively

$$f(x) = x^2\,, \quad f(x) = -x^2\,, \quad f(x) = x^3\,, \quad f(x) = -x^3\,.$$

In each of these examples the critical point is 0 where $f'(0) = 0$. Other good examples are respectively

$$f(x) = x^{\frac{2}{3}}\,, \quad f(x) = -x^{\frac{2}{3}}\,, \quad f(x) = x^{\frac{1}{3}}\,, \quad f(x) = -x^{\frac{1}{3}}\,.$$

In each of these examples f' is undefined at the critical point 0. You should sketch each example over the interval $[-1, 1]$. Your calculator may have problems with the fractional powers for negative values of x, so be careful. You may need to use the fact that $x^{\frac{2}{3}}$ has even symmetry while $x^{\frac{1}{3}}$ has odd symmetry to figure out what those graphs look like for negative values of x.

Example: To determine the monotonicity and extrema of $f(x) = x^2 e^{-x}$ over $(-\infty, \infty)$, first compute $f'(x) = 2xe^{-x} - x^2 e^{-x}$, then factor it as $f'(x) = x(2-x)e^{-x}$. A sign analysis shows that f' is negative over $(-\infty, 0)$, positive over $(0, 2)$, and negative over $(2, \infty)$. From this we can read off that

- f is increasing on $[0, 2]$,
- f is decreasing on $(-\infty, 0]$ and $[2, \infty)$,
- f has a local minimum at 0,
- f has a local maximum at 2.

In addition, you should be able to see (do you?) that $f(x) \to 0$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$. Sketch f with this information and compare it with a calculator graph.

A comparison of critical values is sometimes hard, especially when you are unable to find all the critical points of f. A sign analysis of f' is sometimes cumbersome, especially when you are unable to factor f'. In the case when p is a critical point of f with $f'(p) = 0$, then an alternative route is provided by the **Second Derivative Test for Local Extrema:**

- if $f''(p) > 0$ then p is isolated and f has a local minimum at p;
- if $f''(p) < 0$ then p is isolated and f has a local maximum at p;
- if $f''(p) = 0$ or f'' is undefined at p then there is *no information.*

When we say in the last bullet that there is no information, we mean that *anything can still happen.* To see this, consider the examples

$$f(x) = x^4\,, \quad f(x) = -x^4\,, \quad f(x) = x^3\,, \quad f(x) = -x^3\,, \quad f(x) = x^4 \sin\left(\frac{1}{x}\right),$$

where in the last one we define $f(0) = 0$. In each of these examples $f'(0) = f''(0) = 0$, so that 0 is a critical point and the third bullet applies. They illustrate respectively cases where f has a local minimum at 0, f has a local maximum at 0, f is increasing near 0, f is decreasing near 0, and finally, where 0 is not isolated and f neither has a extrema at nor is monotonic near 0. Other good examples are

$$f(x) = x^{\frac{4}{3}}\,, \quad f(x) = -x^{\frac{4}{3}}\,, \quad f(x) = x^{\frac{5}{3}}\,, \quad f(x) = -x^{\frac{5}{3}}\,, \quad f(x) = x^2 \sin\left(\frac{1}{x}\right).$$

where in the last one we again define $f(0) = 0$. In each of these examples $f'(0) = 0$, while f'' is undefined at 0. Once again they illustrate respectively cases where f has a local minimum at 0, f has a local maximum at 0, f is increasing near 0, f is decreasing near 0, and finally, where 0 is not isolated and f neither has a extrema at nor is monotonic near 0.

Example: To use the Second Derivative Test for Local Extrema to determine the monotonicity and extrema of $f(x) = x^2e^{-x}$ over $(-\infty, \infty)$, first compute $f'(x) = 2xe^{-x} - x^2e^{-x}$, then factor it as $f'(x) = x(2-x)e^{-x}$. The critical points are 0 and 2. Now compute $f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x}$ and evaluate $f''(0) = 2 > 0$ and $f''(2) = -2e^{-2} < 0$. From this we can read off the same information given previously.

Because the Second Derivative Test for Local Extrema generally gives less information than either comparing critical values or the First Derivative Sign Test, and because it incurs the additional cost of taking the second derivative of the function in question, it should only be used when you are unable to do the other two.

2.2: Concavity, Degenerate Points and Inflections. Let I be either (a,b), $[a,b)$, $(a,b]$ or $[a,b]$ for some $a < b$. Recall that if a function f is differentiable over I then we say f is **concave up** over I if f' is increasing over I, and we say f is **concave down** over I if f' is decreasing over I. Accordingly, we say the **concavity** of f over I is either concave up or concave down. Of course, most functions are neither concave up nor concave down over their whole domain. To "determine the concavity of a function" means to identify the intervals over which the function is concave up or concave down. You should be able to do this by studying the sign of its second derivative. For example, we have the following.

Concavity Theorem: If f is continuously differentiable over I and twice differentiable over (a,b), you can read off the following information about f from its second derivative:

- if $f'' > 0$ over (a,b) then f is concave up over I;
- if $f'' < 0$ over (a,b) then f is concave down over I;
- if $f'' = 0$ over (a,b) then f is linear over I.

Reason: Suppose $f'' > 0$ over (a,b). When the Monotonicity Theorem is applied to f', it follows that f' is increasing over I, whereby f is concave up over I. The case where $f'' < 0$ over (a,b) is argued similarly. Now suppose $f' = 0$ over (a,b). When the Monotonicity Theorem is applied to f', it follows that f' is is constant over I. Let $f'(x) = m$ for every x in I. Then set $g(x) = f(x) - mx$ for every x in I. When the Monotonicity Theorem is now applied to g, it follows that g is is constant over I. Let $g(x) = c$ for every x in I. Then $f(x) = mx + g(x) = mx + c$ for every x in I. Hence, f is linear over I. //

We shall say that f has an **upward inflection** at a point p in I if f' has local minimum at p in (a,b). Similarly, we say that f has a **downward inflection** at a point p in I if f' has local maximum at p in (a,b). Points that are either an upward inflection or a downward inflection are called **inflection points** or **inflections** and their corresponding values are called **inflection values**.

To find the inflections of a function f defined over an interval I, the Zero-Slope Theorem applied to f' suggests that we can restrict our search to the critical points of f', namely, to those points p in (a,b) where either $f''(p) = 0$ or f'' is undefined. If p is a point in I where either $f''(p) = 0$ or f'' is undefined then

- p is called a **degenerate point** of f over I;
- $f(p)$ is called a **degenerate value** of f over I.

Sometimes the point $(p, f(p))$ on the graph of f is also called a degenerate point. The intended meaning should be drawn from the context. We will often work with twice differentiable functions, in which case our degenerate points will be of the $f''(p) = 0$ type.

The Concavity Theorem says nothing about degenerate points. It is clear that f cannot change its concavity between successive degenerate points. Given a degenerate point p in the domain of f, we shall say that f is either concave up or concave down **near** p if it is so over the intersection of the domain with some small open interval containing p.

A degenerate point p of a function f is called **isolated** if it is the only degenerate point of f in some small open interval containing p. You should be able to give examples of degenerate points that are isolated and of ones that are not isolated. At an isolated degenerate point p exactly one of only four things can happen; that thing is determined by the **Second Derivative Sign Test for Inflections:**

- if f'' is negative on the left and positive on the right near p
  then f has a upward inflection at p;
- if f'' is positive on the left and negative on the right near p
  then f has a downward inflection at p;
- if f'' is positive on both the left and right near p
  then f is concave up near p;
- if f'' is negative on both the left and right near p
  then f is concave down near p.

Examples that illustrate each of these possibilities are respectively

$$f(x) = x + x^3 \,, \quad f(x) = x - x^3 \,, \quad f(x) = x + x^4 \,, \quad f(x) = x - x^4 \,.$$

In each of these examples the degenerate point is 0 where $f'(0) = 0$. Other good examples are respectively

$$f(x) = x + x^{\frac{5}{3}} \,, \quad f(x) = x - x^{\frac{5}{3}} \,, \quad f(x) = x + x^{\frac{4}{3}} \,, \quad f(x) = x - x^{\frac{4}{3}} \,.$$

In each of these examples f'' is undefined at the degenerate point 0. You should sketch each example over the interval $[-1, 1]$. Your calculator may have problems with the fractional powers for negative values of x, so be careful.

Example: To determine the concavity and inflections of $f(x) = x^2e^{-x}$ over $(-\infty, \infty)$, first compute $f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x}$, then factor it as

$$f''(x) = (x^2 - 4x + 2)e^{-x} = \big((x-2)^2 - 2\big)e^{-x} = \big(x - 2 + \sqrt{2}\big)\big(x - 2 + \sqrt{2}\big)e^{-x}\,.$$

A sign analysis shows that $f''(x)$ is positive on $(-\infty, 2-\sqrt{2})$, negative on $(2-\sqrt{2}, 2+\sqrt{2})$, and positive on $(2+\sqrt{2}, \infty)$. From this we read off that

- f is concave up on $(-\infty, 2-\sqrt{2}]$ and $[2+\sqrt{2}, \infty)$,
- f is concave down on $[2-\sqrt{2}, 2+\sqrt{2}]$,
- f has an upward inflection at $2+\sqrt{2}$,
- f has a downward inflection at $2-\sqrt{2}$.

Combine this information with that on monotonicity and extrema that we obtained earlier and use it to sketch f.

A sign analysis of f'' is sometimes cumbersome, especially when you are unable to factor f''. In the case when p is a degenerate point of f with $f''(p) = 0$, an alternative route is provided by the **Third Derivative Test for Inflection Points:**

- if $f'''(p) > 0$ then p is isolated and f has an upward inflection at p;
- if $f'''(p) < 0$ then p is isolated and f has a downward inflection at p;
- if $f'''(p) = 0$ or f''' is undefined at p then there is *no information.*

When we say in the last bullet that there is no information, we mean that *anything can still happen.* Can you think of examples to illustrate the different possibilities? This test is not mentioned in the book and is seldom used. Indeed, we will not use it.

2.3: Parallels. A careful reading of Sections 2.1 and 2.2 brings out many parallels between them. Below is a table that lists some of these parallels.

Section 2.1	Section 2.2
sign of f'	sign of f''
increasing	concave up
decreasing	concave down
critical point	degenerate point
local minimum	upward inflection
local maximum	downward inflection

3. OPTIMIZATION PROBLEMS

3.1: Some General Guidelines. Optimization problems are problems (usually word problems) whose solution requires you to minimize or maximize some quantity. There is no set of magic steps by which you can solve every such problem. Rather, you should be ready to consider many possible approaches when faced with one. To guide your efforts however, it may be helpful to keep in mind the following three objectives.

1) *Identify the quantity to be extremized and express it in terms of other variables in the problem.* An appropriately labeled picture is often very helpful in this regard. You also should note any constraints on the variables you have introduced. You should be prepared to rethink your choice of variables to help achieve the next objective.
2) *Express the quantity to be extremized as a function of one variable and identify the domain of that variable.* One must use relations between the various variables to eliminate all but one from the expression for the quantity to be extremized. These relations may or may not be stated explicitly in the problem. Similarly, the domain is determined from constraints that may or may not be stated in the problem (like that lengths should be positive).
3) *Solve the resulting problem.* This is usually done by the methods of Section 1.3 of these notes. However, sometimes one needs to think of the problem graphically (such as when extremizing a function of the form $f(x)/x$ where f is given by a graph and not by a formula) or numerically (when exact methods prove too difficult).

These articulate what I call the **IRS** objectives, which can be applied to many types of word problems: **I**dentify the problem; **R**educe the problem to one you can solve; **S**olve the reduced problem.

3.2: Economics Problems. Economics optimization problems have a vocabulary all their own. Once you have mastered it, the problems are no harder than others you will face. There are four basic ideas to keep in mind.

First, in the simple economic models we consider, the **cost** (the outgoing money) to produce a quantity q of goods is given as $C(q)$ by a **cost function** C. This is an increasing function of q that typically has $C(0) > 0$ (to represent start-up costs), is concave down for awhile (as production becomes more efficient), and then becomes concave up (as overtime, expansion, and other costs make production less efficient).

Second, in these models it is assumed that the q goods produced are priced so that they all sell right away. Then the **revenue** (the incoming money) generated by selling these q goods is given as $R(q)$ by a **revenue function** R. If you sell all of these q goods at an identical **price** p then R is simply given by $R(q) = pq$. When p is given as a constant, R is a linear function of q. A more realistic model would have p given as $P(q)$ by a **price function** P that sets the price so that all the q goods produced will be sold right away. In this case one has

$$R(q) = P(q)\,q\,. \tag{3.1}$$

The law of supply and demand suggests that P should be a decreasing function of q.

Third, the **profit** (the money you keep) generated by these q goods is given as $\Pi(q)$ by a **profit function** Π. This is clearly related to the cost and revenue functions by

$$\Pi(q) = R(q) - C(q)\,. \tag{3.2}$$

This relationship is often unstated in economics problems, so you need to know it.

Fourth, the **marginal cost**, **marginal revenue**, and **marginal profit** are respectively defined to be the incremental cost, revenue, and profit generated by an additional good — namely,

$$C(q+1) - C(q)\,, \qquad R(q+1) - R(q)\,, \qquad \Pi(q+1) - \Pi(q)\,. \tag{3.3}$$

When the quantities of goods involved is large, and C, R, and Π are well-behaved functions of q, then q can be considered as a continuous variable and C, R, and Π can be considered to be differentiable functions of q. In this case the marginal cost, marginal revenue, and marginal profit are assumed to be respectively given by

$$C'(q)\,, \qquad R'(q)\,, \qquad \Pi'(q)\,. \tag{3.4}$$

In this setting, 'marginal' is a code word for 'derivative of'.

This vocabulary can be used to express the solution of a problem. For example, if profit is maximized at a critical point of Π, it is clear from (3.2) that at that point

$$C'(q) = R'(q)\,, \tag{3.5}$$

which states that the marginal cost equals the marginal revenue. If C and R are given graphically then (3.5) gives you a way to guess the location of a solution just by looking at the graphs because it shows that at such a point the slope of C and R must be equal.

4. TANGENT LINE APPROXIMATIONS

4.1: Tangent Line Approximations. If f is differentiable at a point c then recall that the tangent line to the curve $y = f(x)$ at c is given by

$$y = f(c) + f'(c)(x - c)\,. \tag{4.1}$$

It is the unique line through the point $(c, f(c))$ with slope $f'(c)$.

The idea of the tangent line approximation is that this line will be a good approximation to the curve $y = f(x)$ so long as x is close to c. Viewed graphically, this idea should seem obvious to you. Another way to understand the tangent line approximation starts with the definition of the derivative at c written in the form

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}\,. \tag{4.2}$$

This can be re-expressed as

$$\lim_{x \to c} \frac{f(x) - f(c) - f'(c)(x - c)}{x - c} = 0\,. \tag{4.3}$$

Because $f(x)-f(c)-f'(c)(x-c)$ is the difference between the value of $f(x)$ and the value of its tangent line approximation at c, this shows that error of the tangent line approximation goes to zero faster than $x - c$ as x approaches c.

The sign of the error made by the tangent line approximation can be determined by analyzing the concavity of f near the point c. Let f be differentiable over an interval I that contains c. If f is concave up over I then

$$f(x) \geq f(c) + f'(c)(x - c) \qquad \text{for every } x \text{ in } I\,. \tag{4.4}$$

Said another way, if f is concave up over I, the tangent line lies below the graph of f over I. On the other hand, if f is concave down over I then

$$f(x) \leq f(c) + f'(c)(x - c) \qquad \text{for every } x \text{ in } I\,. \tag{4.5}$$

Said another way, if f is concave down over I, the tangent line lies above the graph of f over I.

5. ROOT FINDING

5.1: Roots. The roots of a function f are the solutions of the equation $f(x) = 0$. The problem of solving an algebraic equation can always be cast as finding the roots of some function f. For example, if g and h are functions then the solutions of the equation $g(x) = h(x)$ are the roots of the function f given by $f(x) = g(x) - h(x)$.

In seeking the roots of a function you should try to gain an accurate picture of f through a combination of algebraic and graphical analysis. For example, graph f and look for roots. Remember that a graph will only indicate roots that lie within the part of the domain that you have plotted; there may be others. Be sure that you understand the behavior of f outside the part of the domain that you have plotted. Once you have found a root graphically, read off its approximate value, possibly with the help of a zoom feature. If you are trying to solve an equation of the form $g(x) = h(x)$ then it might be easier to graph both g and h and to look for intersections.

Roots may also be approximately located numerically. Consider a function f that is defined over an interval $[a, b]$ for some $a < b$. One may numerically locate a root of f in $[a, b]$ with the following.

Sign Test for Roots: If f is continuous over $[a, b]$ and $f(a)f(b) \leq 0$ then there exists a point p in $[a, b]$ such that $f(p) = 0$.

If $f(a)f(b) = 0$ then either a or b is a root — and maybe both. If $f(a)f(b) < 0$ then f must change its sign somewhere over $[a, b]$. The only way a continuous function can do this is by having a root somewhere in (a, b). This fact seems obvious, but its justification is not as easy as you might guess. We will assume it is true.

To apply the Sign Test for Roots to a function f, first determine the domain of f then break the domain up into intervals over which f is continuous. Finally, seek sign changes of f evaluated at the endpoints of these intervals.

Once you have located a root approximately either by graphical or numerical means, it is best to turn to numerical methods to obtain an accurate value for it. For example, if you have located a root of f in an interval $I = [a, b]$ over which f is continuous, you can refine your knowledge of at least one root by the so-called **Bisection Method:**

- Evaluate f at some point p within I, say the midpoint.
- Use the Sign Test for Roots to determine in which of $[a, p]$ or $[p, b]$ a root lies.
- Replace I with whichever of $[a, p]$ or $[p, b]$ a root lies.
- Repeat until the length of I is within your desired accuracy.

The bisection method will always converge, although we will not prove that here. The endpoints of the interval I provide a lower and upper bound for a root. However, the convergence is quite slow.

5.2: The Newton-Raphson Method. One of the fastest methods to compute roots of a function f is the Newton-Raphson method. Given a guess x_n, we let our next guess x_{n+1} be the x-intercept of the tangent line approximation to f at x_n. In other words, we let x_{n+1} be the solution of

$$f(x_n) + f'(x_n)(x - x_n) = 0\,. \tag{5.1}$$

Provided $f'(x_n) \neq 0$ this can be solved to obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\,. \tag{5.2}$$

The points $\{x_n\}$ so obtained are called the Newton-Raphson iterates. If they converge, they will do so very fast, eventually doubling the number of correct digits with each new iterate.

The Newton-Raphson method works best if a root has been isolated in an interval without critical points. Bounds on the error made by the Newton-Raphson iterates can then be obtained by analyzing the concavity of f near the root. For example, if we denoted the root by x_* then one can see the following.

- If f is increasing and concave up near x_*,
  or is decreasing and concave down near x_*,
  then the sequence $\{x_n\}$ will approach x_* from above.
- If f is increasing and concave down near x_*,
  or is decreasing and concave up near x_*,
  then the sequence $\{x_n\}$ will approach x_* from below.

These observations can also be expressed as follows.

- If $f'(x_*)f''(x_*) > 0$ then the sequence $\{x_n\}$ will approach x_* from above.
- If $f'(x_*)f''(x_*) < 0$ then the sequence $\{x_n\}$ will approach x_* from below.

Hence, the sequence $\{x_n\}$ will always approach x_* from the side on which $f(x)f''(x) > 0$. Do you see why? It is advantageous to make your initial guess on this side.