# LIMITS, CONTINUITY, AND DIFFERENTIABILITY: BASIC CONCEPTS AND GENERAL RULES

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The following is a review of differentiation. It supplements the material we covered both in Chapters 2 and 4 of the book and in the class lectures. It is not my intention that you memorize all of the derivative formulas contained herein, but rather that you familiarize yourself with them. Of course, the ones identified as very important should be memorized. The others can then be easily recovered by the steps indicated in the text. If you are not already, you should become familiar with these steps to the point were they seem obvious to you. When you have reached that stage you will have mastered, rather than have memorized, the formulas. In order to facilitate that process, the formulas are presented in conceptually related groups. The many striking parallels between the formulas for trigonometric and hyperbolic functions should also prove helpful. Mastering these formulas will serve you in two ways. First, it will minimize the number of formulas that you have to memorize now. Second, it will be of enormous value when you study integration later.

# 1. LIMITS AND CONTINUITY

**1.1: Limits.** Calculus is based on the notion of limit. We have already seen this notion arise in different forms when defining the number $e$ and when studying the asymptotic behavior of functions for large $x$. When defining continuity and differentiability, the notion takes the following form.

One first introduces the notion of a **limit point of a set** of real numbers. Specifically, a point $a$ is said to be a limit point of a set $S$ if there are points in $S$ other than $a$ that are arbitrarily close to $a$. For example, if $-\infty < b < c < \infty$ then the set of limit points of the intervals $(b, c)$, $[b, c)$, $(b, c]$, and $[b, c]$ is the closed interval $[b, c]$, while set of limit points of the intervals $(-\infty, b)$ and $(-\infty, b]$ is $(-\infty, b]$, and that of $(c, \infty)$ and $[c, \infty)$ is $[c, \infty)$. Most sets $S$ we will consider will be a finite union of such intervals; in such a case you should be able to identify its limit points. The notion of a limit point can be understood as follows. A point $a$ is a limit point of a set $S$ if there exists a sequence of points $\{x_1, x_2, \cdots\}$ in $S$ that never becomes equal to $a$ but that approaches $a$. For example, 1 is a limit point of the intervals $[0, 1]$ and $[0, 2]$ because $\{.9, .99, .999, \cdots\}$ is a sequence of points in both those intervals that never becomes equal to 1 but that approaches 1.

Next, we consider the notion of the **limit of a function at a point**. Given

- a function $f$ with domain $\mathrm{Dom}(f)$,
- a limit point $a$ of $\mathrm{Dom}(f)$,
- a real number $b$,

you must understand, at least intuitively, what is meant by

$$\lim_{x \to a} f(x) = b\,. \tag{1.1}$$

This is read "the limit of $f(x)$ as $x$ approaches $a$ is $b$". Roughly speaking, this means that $f(x)$ can be made as close as you choose to $b$ by making $x$ sufficiently close but not equal to $a$. If such a number $b$ exists for the given function $f$ and limit point $a$, then the limit of $f$ at $a$ is said to exist; if not, then the limit is said to not exist.

It is important to realize that $x$ plays the role of a "dummy variable" in (1.1). That is to say, we may replace $x$ in (1.1) by any new variable (not $a$, $b$, or $f$) without changing the meaning of (1.1). For example, (1.1) is equivalent to the expressions

$$\lim_{q \to a} f(q) = b\,, \qquad \lim_{t \to a} f(t) = b\,, \qquad \lim_{z \to a} f(z) = b\,.$$

Said another way, (1.1) is a statement about $a$, $b$, and $f$ only.

The notion of limit can be understood numerically as follows. If you pick *any* sequence of points $\{x_1, x_2, \cdots\}$ in $\mathrm{Dom}(f)$ that approaches $a$ while never equaling $a$ then the corresponding sequence of values $\{f(x_1), f(x_2), \cdots\}$ approaches $b$. You can employ this understanding to explore the existence of a limit with your calculator. For example, you can usually select one such sequence $\{x_1, x_2, \cdots\}$ truncated at a finite number of points and see if the corresponding values $\{f(x_1), f(x_2), \cdots\}$ seem to be approaching a value $b$. There is however a big potential weaknesses of this strategy — namely, strictly speaking you should check *every infinite* sequence of points that approaches $a$ while never equaling $a$. Your numerical evidence can therefore be misleading (as some of you may discover on the homework).

The notion of limit can be understood *graphically* in terms of windows on your calculator as follows. Given any $y$-interval about $b$ of the form $[b-\epsilon, b+\epsilon]$ for some positive $\epsilon$, you can find an $x$-interval about $a$ of the form $[a-\delta, a+\delta]$ for some positive $\delta$ such that the graph of $f$ lies below the top and above the bottom of the window with $x$-values in $[a-\delta, a+\delta]$ and $y$-values in $[b-\epsilon, b+\epsilon]$. You can employ this understanding to explore the existence of a limit with your calculator. For example, you can select a single $y$-interval $[b-\epsilon, b+\epsilon]$ for some small positive $\epsilon$, and then play with the $x$-interval to see if you can find a $\delta$ small enough that the graph of $f$ lies below the top and above the bottom of the window with $x$-values in $[a-\delta, a+\delta]$. There is however a big potential weaknesses of this strategy — namely, strictly speaking you should check that this can be done for *every* $y$-interval $[b-\epsilon, b+\epsilon]$. Such graphical evidence can therefore sometimes be misleading.

Difficulties such as those described above are avoided by making precise definitions of both limit points of sets and limits of functions (1.1). For example, $a$ is said to be a limit point of a set $S$ if given any distance $\delta$ you can find a point $x$ in $S$ that is not $a$ yet within $\delta$ of $a$ — i.e. you can find a point $x$ in $S$ such that

$$0 < |x-a| < \delta\,. \tag{1.2}$$

Then (1.1) means that given any distance $\epsilon$ you can find a distance $\delta$ for which you can show that $f(x)$ is within $\epsilon$ of $b$ whenever $x$ is in $\mathrm{Dom}(f)$ and within $\delta$ of but not equal to $a$ — i.e. you can show that

$$|f(x)-b| < \epsilon \qquad \text{whenever} \qquad x \in \mathrm{Dom}(f) \text{ and } 0 < |x-a| < \delta\,. \tag{1.3}$$

At this stage in your study of calculus it is not necessary for you to understand or even memorize these more precise definitions. After all, calculus existed for over a century and a half before the need for such precise definitions was evident. An intuitive understanding of the notion of limit was however central right from the start. Similarly, you must build such an understanding to get started in your study of calculus. If you keep up your study, you too will eventually see the need for more precise definitions.

**1.2: One-Sided Limits.** Another notion is that of a **one-sided limit of a function at a point**. Given

- a function $f$ with domain $\mathrm{Dom}(f)$,
- a point $a$ that is a limit point of $\mathrm{Dom}(f) \cap (a,\infty)$ $\big(\text{or } \mathrm{Dom}(f) \cap (-\infty,a)\big)$,
- a number $b$,

you must understand, at least intuitively, what is meant respectively by

$$\lim_{x\to a^+} f(x) = b\,, \qquad \left(\text{or } \lim_{x\to a^-} f(x) = b\right). \tag{1.4}$$

This is read "the right-hand (left-hand) limit of $f(x)$ as $x$ approaches $a$ is $b$". Roughly speaking, this means that $f(x)$ can be made as close as you choose to $b$ by making $x > a$ (or $x < a$) sufficiently close to $a$. As before, if such a number $b$ exists for the given function $f(x)$ and limit point $a$, then the limit of $f$ at $a$ is said to exist; if not, then the limit is said to not exist. The notion of a right-hand (left-hand) limit can also be understood numerically as follows. If you pick *any* sequence of points $\{x_1, x_2, \cdots\}$ to the right (left) of $a$ and in $\mathrm{Dom}(f)$ that approaches $a$, then the corresponding sequence of values $\{f(x_1), f(x_2), \cdots\}$ approaches $b$. Of course, a precise definition of the one-sided limits (1.4) can be made in the spirit of (1.3), but we will not do so here.

Roughly speaking, limit of $f(x)$ as $x$ approaches $a$ exists when the right-hand and left-hand limits of $f(x)$ as $x$ approaches $a$ both exist and are equal. More precisely, if $a$ is a limit point of both $\mathrm{Dom}(f) \cap (a, \infty)$ and $\mathrm{Dom}(f) \cap (-\infty, a)$ then

$$\lim_{x \to a} f(x) = b\,,$$

if and only if

$$\lim_{x \to a^+} f(x) = b\,, \quad \text{and} \quad \lim_{x \to a^-} f(x) = b\,.$$

If the right-hand and left-hand limits of $f(x)$ as $x$ approaches $a$ both exist but are not equal then the graph of $f$ will exhibit a jump across $a$.

**1.3: Limits of Combinations of Functions.** One may combine two functions $f$ and $g$ algebraically. For example, let $a$, $b$ and $c$ be numbers such that

$$\lim f(x) = b\,, \qquad \text{and} \qquad \lim g(x) = c\,, \tag{1.5}$$

where here "lim" stands for one of

$$\lim_{x \to a}\,, \qquad \lim_{x \to a^+}\,, \qquad \lim_{x \to a^-}\,. \tag{1.6}$$

Then we have the **sum, product, and quotient limit rules:**

$$\begin{aligned} \lim \big(f(x) \pm g(x)\big) &= b \pm c\,, \\ \lim \big(f(x)\, g(x)\big) &= b\, c\,, \\ \lim \frac{f(x)}{g(x)} &= \frac{b}{c} \qquad \text{provided } c \neq 0\,. \end{aligned} \tag{1.7}$$

Limits of more complicated algebraic combinations can be built up from these.

One may also combine two functions $f$ and $g$ by composition. We shall denote by $g(f)$ the function whose value is given by $g(f(x))$ for every point $x$ in the domain defined by $\mathrm{Dom}(g(f)) \equiv \{x \in \mathrm{Dom}(f) : f(x) \in \mathrm{Dom}(g)\}$. We now let $a$ be a limit point of $\mathrm{Dom}(g(f))$ and consider two situations. First, if $b$ is a number such that

$$\lim_{x \to a} f(x) = b, \qquad \text{and} \qquad \lim_{y \to b} g(y) = g(b), \tag{1.8}$$

then we have the **composition limit rule:**

$$\lim_{x \to a} g(f(x)) = g\Big(\lim_{x \to a} f(x)\Big) = g(b). \tag{1.9}$$

Second, if

$$\lim_{x \to a} f(x) = f(a), \tag{1.10}$$

and $f$ is either increasing or decreasing over an open interval containing $a$, then we have the **change of variable limit rule:**

$$\lim_{y \to f(a)} g(y) = \lim_{x \to a} g(f(x)). \tag{1.11}$$

Such a simple rules do not generally hold for one-sided limits.

**1.4: Continuity.** A function $f$ is said to be **continuous** at a point $a$ in $\mathrm{Dom}(f)$ if either $a$ is not a limit point of $\mathrm{Dom}(f)$ or

$$\lim_{x \to a} f(x) = f(a). \tag{1.12}$$

Here (1.12) is asserting two things:

- the limit on the left side of (1.12) exists;
- the limit equals $f(a)$.

You should know examples of functions that fail to be continuous at a point in its domain both where the limit on the left of (1.12) fails to exist and where the limit exists but does not equal $f(a)$. You should be able to tell by looking at the graph of a function where it is continuous.

It follows from the sum, product and quotient limit rules (1.7) that if $f$ and $g$ are functions that are both continuous at the point $a$ then the functions $f \pm g$ and $f\,g$ will be continuous at the point $a$, as will the function $f/g$ provided $g(a) \neq 0$. Moreover, the composition limit rule (1.9) shows that if $f$ continuous at the point $a$ while $g$ is continuous at the point $f(a)$ then the composition $g(f)$ is continuous at the point $a$.

A function that is continuous at every point in an interval is said to be continuous over that interval. Roughly speaking, when drawing the graph of such a function $f$ over such an interval, one need not lift the pen or pencil from the paper. This is because (1.12) states that as the pen moves along the graph $(x, f(x))$ it will approach the point $(a, f(a))$ as $x$ tends to $a$. The graph of $f$ will consequently have no breaks, jumps, or holes over the interval.

A function that is continuous at every point in its domain is said to be continuous. Every elementary function is continuous.

## 2. BASICS OF DIFFERENTIATION

**2.1: Differentiability.** Given any function $f$, the slope of the secant line through any two points $(a, f(a))$ and $(b, f(b))$ on its graph is given by

$$\frac{f(b) - f(a)}{b - a} . \tag{2.1}$$

This quantity is called a **difference quotient**. It is defined over all points $a$ and $b$ in $\mathrm{Dom}(f)$ for which $b \neq a$. It is undefined when $b = a$.

A function $f$ is said to be **differentiable** at a point $a$ in $\mathrm{Dom}(f)$ whenever

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a} \quad \text{exists} , \tag{2.2}$$

This will be the case whenever a unique tangent line to the graph at $(a, f(a))$ exists and is not vertical, in which case the **slope of the tangent line** is given by

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} , \tag{2.3}$$

whereby the equation of the tangent line is given by

$$y = f(a) + f'(a)(x - a) . \tag{2.4}$$

By replacing $b$ by $a + h$ in (2.3), the slope of this tangent line may be expressed as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} . \tag{2.5}$$

It should be evident to you that (2.5) is completely equivalent to (2.3). Visually, if the graph of a function $f$ at $(a, f(a))$ either has no unique tangent line or has a vertical tangent line then $f$ is not differentiable at the point $a$. A function that is differentiable at every point in an interval is said to be differentiable over that interval. You should be able to tell by looking at the graph of a function where it is differentiable. Can you see that the functions $|x|$ and $x^{1/3}$ are not differentiable at 0 for different reasons?

It is easy to see that if $f$ is differentiable at the point $a$ then it is continuous at $a$. Indeed, for every $b$ in $\mathrm{Dom}(f)$ such that $b \neq a$ one has the identity

$$f(b) = f(a) + \frac{f(b) - f(a)}{b - a} (b - a) . \tag{2.6}$$

If we let $b$ approach $a$ in (2.6) then because $f$ is differentiable at $a$ one sees that

$$\begin{aligned} \lim_{x \to a} f(x) = \lim_{b \to a} f(b) &= f(a) + \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \cdot \lim_{b \to a} (b - a) \\ &= f(a) + f'(a) \cdot 0 = f(a) , \end{aligned}$$

whereby $f$ is continuous at $a$. The converse is not true. Indeed, there are functions that are continuous everywhere yet differentiable nowhere. The construction of such examples is beyond the scope of this course. You should however be able to give examples of functions that are continuous but not differentiable at some point. Both examples given at the end of the last paragraph are continuous at 0.

To test of your understanding, consider the functions $f$ and $g$ given by

$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ x\sin(1/x) & \text{otherwise}, \end{cases} \qquad g(x) = \begin{cases} 0 & \text{for } x = 0 \\ x^2\cos(1/x) & \text{otherwise}. \end{cases}$$

Can you see that

1) $f$ and $g$ are even?
2) $f$ oscillates between the lines $y = x$ and $y = -x$ near zero?
3) $g$ oscillates between the parabolas $y = x^2$ and $y = -x^2$ near zero?
4) $f$ has an horizontal asymptote of $y = 1$?
5) $g$ behaves like $x^2$ for large values of $|x|$?
6) $f$ and $g$ are continuous at $x = 0$?
7) $f$ is not differentiable at $x = 0$?
8) $g$ is differentiable at $x = 0$ with $g'(0) = 0$?

WARNING: Your calculator may not do a good job of showing the behavior of these functions near zero.

**2.2: Derivatives.** The derivative of a function $f$, which is defined at every point $x$ where $f$ is differentiable, is the function whose value at $x$ is the slope of the tangent line to the graph of $f$ at $x$. Hence, by (2.5) the derivative of $f$ at $x$ is given by

$$f'(x) = \frac{d}{dx}f(x) \equiv \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{2.7}$$

The second derivative of $f$ is the derivative of its derivative. It is defined by

$$f''(x) = \frac{d^2}{dx^2}f(x) \equiv \frac{d}{dx}\left(\frac{d}{dx}f(x)\right). \tag{2.8}$$

In a similar way the $n^{th}$ derivative of $f$ is defined by

$$f^{(n)}(x) = \frac{d^n}{dx^n}f(x) \equiv \frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}}f(x)\right). \tag{2.9}$$

If $f$ has all its derivatives at a point $a$, it is said to be infinitely differentiable at $a$.

If the variable $y$ is a function of the variable $x$ then we will sometimes denote the first, second, and $n^{th}$ derivatives of this function by

$$\frac{dy}{dx}, \qquad \frac{d^2y}{dx^2}, \qquad \text{and} \quad \frac{d^ny}{dx^n}. \tag{2.10}$$

There are many other commonly used notations for derivatives. (You may even have seen a few others already.) Such a variety is not too surprising once you realize that derivatives are among the most useful objects in all of mathematics.

You should be able to determine information about the graph of a function from its derivatives. For example, let $f$ be continuous over an interval $I$, where $I$ is either $(a,b)$, $[a,b)$, $(a,b]$ or $[a,b]$ for some $a<b$. Then if $f$ is differentiable over $(a,b)$, you can read off the following information about the graph of $f$ from its first derivative:

- if $f'>0$ over $(a,b)$ then $f$ is increasing over $I$;
- if $f'<0$ over $(a,b)$ then $f$ is decreasing over $I$;
- if $f'=0$ over $(a,b)$ then $f$ is constant over $I$.

Moreover, if $f$ is twice differentiable over $(a,b)$, you can read off the following information about the graph of $f$ from its second derivative:

- if $f''>0$ over $(a,b)$ then $f$ is concave up over $I$;
- if $f''<0$ over $(a,b)$ then $f$ is concave down over $I$;
- if $f''=0$ over $(a,b)$ then $f$ is linear over $I$.

Given the graph of a function you should also be able to roughly sketch the graph of its first two derivatives based on these facts.

**2.3: Interpretations of Derivatives.** When a function $f$ is defined over an interval $[a,b]$ with $a<b$, then the difference quotient (2.1) can be understood as the average change of $f(x)$ with respect to $x$ over $[a,b]$. Then whenever it exists, $f'(a)$ can be understood as the rate of change of $f(x)$ with respect to $x$ at $a$. Because the difference quotients clearly have units equal to the units of $f(x)$ divided by the units of $x$, the same is true of $f'(x)$ as it is the limit of such difference quotients.

For example, if $s(t)$ gives the height in meters at time $t$ in seconds of an object moving vertically, then $s'(t)$ gives the rate height changes with respect to time (i.e. the vertical velocity) in meters per second at time $t$. Similarly, if $v(t)$ gives the vertical velocity in meters per second as a function of time $t$ in seconds of an object moving vertically, then $v'(t)$ gives the rate velocity changes with respect to time (i.e. the acceleration) in meters per second per second at time $t$.

Finally, if $f(p)$ gives the number of widgets sold by a company as a function of the price of a widget $p$ in dollars then $f'(p)$ gives the rate in widgets per dollar that sales will change with respect to changes in the price of a widget. So that $f(5)=500,000$ means you sell $500,000$ widgets at a price of 5 dollars each, while $f'(5)=-80,000$ means sales would decrease at a rate of $80,000$ widgets per dollar as you raise the price. In particular, if you raise the price 50 cents then sales would decrease by about $40,000$ to about $460,000$ widgets. Do you see why this may be a good thing for the company to do?

**2.4: Basic Derivatives from the Definition.** There are a few basic functions whose derivative formulas you should be able to derive directly from the definition (2.7). These include

$$\begin{aligned}
&\frac{d}{dx}\,1 = 0\,, && \frac{d}{dx}\,x^n = nx^{n-1}\,, \\
&\frac{d}{dx}\,e^x = e^x\,, && \frac{d}{dx}\,\ln(x) = \frac{1}{x}\,, \\
&\frac{d}{dx}\,a^x = \ln(a)a^x\,, && \frac{d}{dx}\,\log_a(x) = \frac{1}{\ln(a)}\,\frac{1}{x}\,, \\
&\frac{d}{dx}\,\sin(x) = \cos(x)\,, && \frac{d}{dx}\,\cos(x) = -\sin(x)\,, \\
&\frac{d}{dx}\,\sinh(x) = \cosh(x)\,, && \frac{d}{dx}\,\cosh(x) = \sinh(x)\,,
\end{aligned} \tag{2.11}$$

and any simple variants thereof. The top two are straightforward. The first is trivial, and when $n$ is an integer the second only requires simple algebraic manipulation of the difference quotient before passing to the limit. For example, when $n$ is a positive integer you have to expand $(x+h)^n$ by the binomial formula. You should be comfortable with cases in which $n$ is a positive or negative integer whose absolute value is not too large.

The formulas for the expontential and logarithmic derivatives are derived using the fact, which you should know, that the number $e$ is given by the limit

$$e = \lim_{s\to 0}(1+s)^{1/s}\,. \tag{2.12}$$

Given this limit, you should be able to obtain the derivative formulas for logarithms. You should also be able to use (2.12) and the change of variable limit rule (1.10) with $s = a^h - 1$ to derive the limit

$$\lim_{h\to 0}\frac{a^h - 1}{h} = \ln(a)\,. \tag{2.13}$$

From this you should be able to obtain derivative formulas for the exponentials.

The formulas for the sine and cosine derivatives are obtained through the appropriate trigonometric addition formulas and the limits

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1\,, \qquad \lim_{h\to 0}\frac{1-\cos(h)}{h} = 0\,. \tag{2.14}$$

The first limit was argued in class by comparing the area of a pizza pie slice of angle $h$ with that of a larger and a smaller triangle. This led us to the inequalities

$$\cos(h) \leq \frac{\sin(h)}{h} \leq 1 \qquad \text{for every } |h| < \tfrac{\pi}{2}\,. \tag{2.15}$$

Given this inequality, it is easy to obtain the first limit. Given the first limit, you should be able to obtain the second limit. Given both limits, you should be able to derive the sine and cosine derivative formulas.

Finally, the formulas for the sinh and cosh derivatives can be obtained through the appropriate hyperbolic addition formulas and the limits

$$\lim_{h \to 0} \frac{\sinh(h)}{h} = 1, \qquad \lim_{h \to 0} \frac{\cosh(h) - 1}{h} = 0, \tag{2.16}$$

Given the limit (2.13), you should be able to obtain the first limit. Given the first limit, you should be able to obtain the second limit. Given both limits, you should be able to derive the sinh and cosh derivative formulas. Alternatively, the derivative formulas could be derived by first using their definitions to express $\sinh(x) = (e^x - e^{-x})/2$ and $\cosh(x) = (e^x + e^{-x})/2$ and then proceeding as for the expontential derivative formula. The key once again will be the limit (2.13). I suggest that you know both approaches.

**1.6: Shifts, Stretches, Flips, Symmetries, and Derivatives.** Given a differentiable function $f$, it can be seen easily from the definition of the derivative (2.7) that

$$\begin{aligned} \frac{d}{dx}\big(f(x) + b\big) &= f'(x), \\ \frac{d}{dx}\big(kf(x)\big) &= k\, f'(x), \\ \frac{d}{dx}\big(-f(x)\big) &= -f'(x). \end{aligned} \tag{2.17}$$

You should be able to visualize these relations in terms of graphs. They state that a vertical shift does not change the derivative of a function, while a vertical stretch or flip changes the derivative in the same way. It can also be seen from definition (2.7) that

$$\begin{aligned} \frac{d}{dx} f(x - a) &= f'(x - a), \\ \frac{d}{dx} f(x/m) &= \frac{1}{m}\, f'(x/m), \\ \frac{d}{dx} f(-x) &= -f'(-x). \end{aligned} \tag{2.18}$$

These relations could also have been derived using the chain rule. Graphically, they state that a horizontal shift changes the derivative in the same way, a horizontal stretch by $m$ changes the derivative by both a horizontal stretch by $m$ and a vertical stretch by $1/m$, while a horizontal flip changes the derivative by both a horizontal and a vertical flip.

Given (2.17) and (2.18), you should be able to show the symmetry relations:

- if $f$ is even then $f'$ is odd;
- if $f$ is odd then $f'$ is even;
- if $f$ is periodic with period $p$ then so is $f'$;
- if $f$ is antiperiodic with antiperiod $p$ then so is $f'$.

Once again, you should be able to visualize these relations in terms of graphs.

## 3. GENERAL RULES FOR DIFFERENTIATION

**3.1: Rules for Linear Combinations of Functions.** Given any two differentiable functions $u$ and $v$, and constant $k$, the functions $ku$ and $u+v$ are also differentiable and their derivatives are given by the so-called **multiplication rule** and **sum rule:**

$$\frac{d}{dx}(ku) = k\,\frac{du}{dx}\,, \qquad \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}\,. \tag{3.1}$$

These are expressed in words as "the derivative of a multiple is the multiple of the derivative" and "the derivative of a sum is the sum of the derivative" respectively. These rules follow from the definition of the derivative (2.7) and the algebraic identities

$$\begin{aligned} \frac{ku(x+h) - ku(x)}{h} &= k\,\frac{u(x+h) - u(x)}{h}\,, \\ \frac{u(x+h) + v(x+h) - u(x) - v(x)}{h} &= \frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\,. \end{aligned}$$

The multiplication and sum rules (3.1) are used all the time. They are easy to master.

The linear combinations of $n$ given functions $\{u_1, u_2, \cdots, u_n\}$ are all those functions of the form $k_1u_1 + k_2u_2 + \cdots + k_nu_n$ for some choice of $n$ constants $\{k_1, k_2, \cdots, k_n\}$. In other words, the linear combinations are all those function that can be built up from the given functions $\{u_1, u_2, \cdots, u_n\}$ by repeated multiplication by constants and addition. If each of the given functions $\{u_1, u_2, \cdots, u_n\}$ is differentiable then repeated applications of the multiplication and sum rules (3.1) show that each such linear combination is also differentiable and its derivative is given by the **linear combination rule:**

$$\frac{d}{dx}\big(k_1u_1 + k_2u_2 + \cdots + k_nu_n\big) = k_1\frac{du_1}{dx} + k_2\frac{du_2}{dx} + \cdots + k_n\frac{du_n}{dx}\,. \tag{3.2}$$

This is expressed in words as "the derivative of a linear combination is the linear combination of the derivatives". It is important to understand that this rule need not be memorized because all it does is embody repeated applications of (3.1). That is to say, if you have truly mastered (3.1) then (3.2) will seem obvious to you and will not need to be memorized.

Given any two differentiable functions $u$ and $v$, a particular instance of the linear combination rule (3.2) is the **difference rule:**

$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}\,. \tag{3.3}$$

This can be expressed as "the derivative of a difference is the difference of the derivatives". As an instance of (3.2), this rule also should seem obvious based on a mastery of (3.1).

**3.2: Rules for Algebraic Combinations of Functions.** Given any two differentiable functions $u$ and $v$, the function $uv$ is also differentiable and its derivative is given by the so-called **product (or Leibnitz) rule:**

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}\,. \tag{3.4}$$

This is not as simple to express in words as say the sum rule, but may be rendered as "the derivative of a product is the derivative of the first times the second plus the first times the derivative of the second". This rule follows directly from the definition and the algebraic identity

$$\frac{u(x+h)v(x+h) - u(x)v(x)}{h} = \frac{u(x+h) - u(x)}{h}\, v(x+h) + u(x)\, \frac{v(x+h) - v(x)}{h}\,.$$

The product rule is a very important general rule for differentiation. You must master it. In fact, all the other rules in this section will essentially follow from the product rule.

If one considers the product of three differentiable functions $u$, $v$, and $w$ then two applications of (3.4) show that

$$\frac{d}{dx}(uvw) = \frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx}\,.$$

More generally, given $n$ differentiable functions $\{u_1, u_2, \cdots, u_n\}$, their product $u_1 u_2 \cdots u_n$ is differentiable and its derivative is given by the **general Leibnitz rule:**

$$\frac{d}{dx}(u_1 u_2 \cdots u_n) = \frac{du_1}{dx}u_2 \cdots u_n + u_1\frac{du_2}{dx} \cdots u_n + \cdots + u_1 u_2 \cdots \frac{du_n}{dx}\,. \tag{3.5}$$

It is important to understand that this rule need not be memorized because all it does is embody repeated applications of (3.4). That is to say, if you have truly mastered (3.4) then (3.5) will seem obvious to you and will not need to be memorized.

A consequence of setting $v = 1/u$ in the product rule (3.4) is the **reciprocal rule:**

$$\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx} \qquad \text{wherever } u \neq 0\,. \tag{3.6}$$

If the reciprocal rule is combined with the product rule then you obtain the **quotient rule:**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2} \qquad \text{wherever } v \neq 0\,. \tag{3.7}$$

This rule is more complicated to express in words than the product rule, but may be rendered as "the derivative of a quotient is the derivative of the top times the bottom minus the top times the derivative of the bottom, all over the bottom squared", or more poetically, "bottom-dee-top minus top-dee-bottom over bottom squared". While it is very helpful to have this rule memorized, it is not critical. In every instance that the quotient rule can be applied, the quotient can be recast as a product to which the product rule (3.4) can be applied. That is after all how the quotient rule was derived above.

If the general Leibnitz rule (3.5) is specialized to the case where all the functions $u_k$ are the same function $u$ then it reduces to the **monomial power rule:**

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}\,. \tag{3.8}$$

The monomial power rule was derived above for positive integers $n$. When it is combined with the reciprocal rule (3.6), one sees that it extends to negative integers $n$. This rule can be extended further. Namely, given any differentiable function $u$ and any rational number $p$ for which $u^p$ is defined, the function $u^p$ is differentiable wherever $u^{p-1}$ is defined and its derivative is given by the **rational power rule:**

$$\frac{d}{dx}u^p = pu^{p-1}\frac{du}{dx}\,. \tag{3.9}$$

Wherever $u \neq 0$ this rule can be derived as follows. Because $p$ is rational it can be expressed as $p = m/n$ where $m$ and $n$ are integers and $n > 0$. If the monomial power rule (3.8) is then applied to each side of the identity $(u^p)^n = u^m$, one finds that

$$n(u^p)^{n-1}\frac{d}{dx}u^p = mu^{m-1}\frac{du}{dx}\,,$$

which is equivalent to the rational power rule wherever $u \neq 0$. Points where $u = 0$ and $p \geq 1$ can be treated directly from the definition of the derivative.

**3.3: Rules for Compositions of Functions.** Given two differentiable functions $v$ and $u$, the derivative of their composition $v(u)$ is given by the **chain rule:**

$$\frac{d}{dx}v(u) = v'(u)\,\frac{du}{dx}\,. \tag{3.10}$$

This is also tricky to express in words, but may be rendered as "the derivative of a composition is the derivative of the outer, evaluated at the inner, times the derivative of the inner". You could also say "the derivative of a composition is the product of the derivatives", provided you realize that this leaves a lot unsaid about the arguments of the derivatives involved. If the functions $u$ and $v$ relate the variables $x$, $y$, and $z$ by $z = v(y)$ and $y = u(x)$, then (3.10) may be expressed as

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}\,. \tag{3.11}$$

The chain rule is the most important general rule for differentiation. You must master it.

It is natural to think that the chain rule can be derived by letting $h$ go to zero in the algebraic identity

$$\frac{v(u(x+h)) - v(u(x))}{h} = \frac{v(u(x+h)) - v(u(x))}{u(x+h) - u(x)} \, \frac{u(x+h) - u(x)}{h} \, .$$

However, this argument does not work because the identity breaks down wherever the $u(x+h) - u(x)$ that appears in the denominator becomes zero. This difficulty is overcome by observing that if $v$ is differentiable at a point $b$ then a **continuous difference quotient** may be defined for every $y$ in $\mathrm{Dom}(v)$ by

$$Q_b v(y) \equiv \begin{cases} \dfrac{v(y) - v(b)}{y - b} & \text{for } y \neq b \, , \\ v'(b) & \text{for } y = b \, . \end{cases}$$

This is a continuous function of $y$ at $b$ and satisfies

$$v(y) - v(b) = Q_b v(y) \, (y - b) \, .$$

Now set $b = u(x)$ and $y = u(x+h)$ in this relation and divide by $h$ to obtain

$$\frac{v(u(x+h)) - v(u(x))}{h} = Q_{u(x)} v\big(u(x+h)\big) \, \frac{u(x+h) - u(x)}{h} \, .$$

The chain rule (3.9) then follows from the compostion limit rule (1.9) and the definition of the derivative (2.7) by letting $h$ go to zero.

If one considers the composition of three differentiable functions, $w$, $v$, and $u$, then two applications of (3.10) show that

$$\frac{d}{dx} w(v(u)) = w'(v(u)) \, v'(u) \, \frac{du}{dx} \, .$$

More generally, if one considers $n$ differentiable functions $\{u_1, u_2, \cdots, u_n\}$, then $n-1$ applications of (3.9) show their composition $u_1(u_2(u_3(\cdots(u_n)\cdots)))$ is differentiable and its derivative is given by the **linked chain rule:**

$$\frac{d}{dx} u_1(u_2(u_3(\cdots(u_n)\cdots))) = u_1'(u_2(u_3(\cdots(u_n)\cdots))) \, u_2'(u_3(\cdots(u_n)\cdots)) \cdots \frac{du_n}{dx} \, . \tag{3.12}$$

If the functions $\{u_1, u_2, \cdots, u_n\}$, relate the variables $\{y_1, y_2, \cdots, y_n\}$ and $x$ by $y_1 = u_1(y_2)$, $y_2 = u_2(y_3)$, $\cdots$, and $y_n = u_n(x)$, then (3.11) may be expressed as

$$\frac{dy_1}{dx} = \frac{dy_1}{dy_2} \frac{dy_2}{dy_3} \cdots \frac{dy_n}{dx} \, . \tag{3.13}$$

It is important to understand that this rule need not be memorized because all it does is embody repeated applications of (3.10). That is to say, if you have truly mastered (3.10) then (3.12) will seem obvious to you and will not need to be memorized.

**3.4: Rules for Some Transcendental Combinations of Functions.** If the exponential rule of (2.11) is combined with the chain rule (3.10) you obtain the **exponential rules:**

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}, \qquad \frac{d}{dx}a^u = \ln(a)\, a^u \frac{du}{dx}. \tag{3.14}$$

The first of these is the most important and should be known. The second is easily recovered from the first by employing the fact that $a^u = e^{\ln(a)u}$. Similarly, if the logarithmic rule of (2.11) is combined with the chain rule (3.10) then for any positive function $u$ one obtains the **logarithmic rules:**

$$\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}, \qquad \frac{d}{dx}\log_a(u) = \frac{1}{\ln(a)}\frac{1}{u}\frac{du}{dx}. \tag{3.15}$$

Once again the first of these is the most important and should be known. The second is easily recovered from the first by employing the fact that $\log_a(u) = \ln(u)/\ln(a)$.

Given any positive function $u$ and any real constant $p$, one now has the **real power rule:**

$$\frac{d}{dx}u^p = p\, u^{p-1}\frac{du}{dx}. \tag{3.16}$$

This rule is a special case of the rule for general power functions. It can be derived by differentiating the identity $\log(u^p) = p\ln(u)$ using the first logarithmic rule (3.15) on each side. Alternatively, it can be derived by differentiating the identity $u^p = e^{p\ln(u)}$ using the first exponential rule (3.14) followed by the first logarithmic rule (3.15). Try both ways to find the one with which you are most comfortable.

The first way mentioned above to derive the real power rule is an application of a useful method called logarithmic differentiation. It is most useful if you have to differentiate a product of the form $u_1^{p_1}u_2^{p_2}\cdots u_n^{p_n}$. By taking the logarithm of this product, you obtain

$$\ln\big(u_1^{p_1}u_2^{p_2}\cdots u_n^{p_n}\big) = p_1\ln(u_1) + p_2\ln(u_2) + \cdots p_n\ln(u_n).$$

Repeated use of the first logarithmic rule (3.15) then gives the **logarithmic differentiation rule:**

$$\frac{d}{dx}\big(u_1^{p_1}u_2^{p_2}\cdots u_n^{p_n}\big) = u_1^{p_1}u_2^{p_n}\cdots u_n^{p_n}\left(\frac{p_1}{u_1}\frac{du_1}{dx} + \frac{p_2}{u_2}\frac{du_2}{dx} + \cdots + \frac{p_n}{u_n}\frac{du_n}{dx}\right). \tag{3.17}$$

This should not be memorized as a rule, but rather mastered as a method. It is simply a way to compactly organize the job of taking the derivative of a complicated product.

Any positive function $u$ may be exponentiated to the power of any other function $v$ to obtain $u^v$. If $u$ and $v$ are differentiable then so is $u^v$, and it derivative is given by the **combined exponential-power rule:**

$$\frac{d}{dx}\big(u^v\big) = v\, u^{v-1}\frac{du}{dx} + \ln(u)\, u^v \frac{dv}{dx}\,. \tag{3.18}$$

This rule need not be memorized as it is very easily recovered by logarithmic differentiation. One differentiates the identity $\ln(u^v) = \ln(u)v$ using the first logarithmic rule (3.15) on $\ln(u^v)$ followed by the product rule (3.4) on $\ln(u)v$ and the first logarithmic rule (3.15) on $\ln(u)$. Alternatively, one could differentiate the identity $u^v = e^{\ln(u)v}$ using the first exponential rule (3.14) followed by the product rule (3.4) on $\ln(u)v$ and the first logarithmic rule (3.15) on $\ln(u)$. Again, try both ways to find the one with which you are most comfortable.

**3.5: Rules for Trigonometric and Hyperbolic Functions.** If the sin and cos rules of (2.11) are combined with the chain rule (3.10) you obtain the **sin and cos rules:**

$$\frac{d}{dx}\sin(u) = \cos(u)\frac{du}{dx}\,, \qquad \frac{d}{dx}\cos(u) = -\sin(u)\frac{du}{dx}\,. \tag{3.19}$$

You should know these formulas. If the quotient and reciprocal rules are then applied to the definitions of the other trigonometric functions in terms of sin and cos, one obtains the other **trigonometric rules:**

$$\begin{aligned} &\frac{d}{dx}\tan(u) = \sec^2(u)\frac{du}{dx}\,, && \frac{d}{dx}\cot(u) = -\csc^2(u)\frac{du}{dx}\,, \\ &\frac{d}{dx}\sec(u) = \sec(u)\tan(u)\frac{du}{dx}\,, && \frac{d}{dx}\csc(u) = -\csc(u)\cot(u)\frac{du}{dx}\,. \end{aligned} \tag{3.20}$$

It is best if you know these formulas too. If you should forget any one of them however, you should be able to recover it as outlined above.

If the sinh and cosh rules of (2.11) are combined with the chain rule (3.10) you obtain the **sinh and cosh rules:**

$$\frac{d}{dx}\sinh(u) = \cosh(u)\frac{du}{dx}\,, \qquad \frac{d}{dx}\cosh(u) = \sinh(u)\frac{du}{dx}\,. \tag{3.21}$$

It is best if you know these formulas. If you should forget either one of them however, you should be able to recover it from the first exponential formula (3.14) and the definitions of sinh and cosh in terms of exponential:

$$\sinh(u) = \frac{e^u - e^{-u}}{2}\,, \qquad \cosh(u) = \frac{e^u + e^{-u}}{2}\,.$$

If the quotient and reciprocal rules are then applied to the definitions of the other hyperbolic functions in terms of sinh and cosh, one obtains the other **hyperbolic rules:**

$$\begin{aligned}
\frac{d}{dx}\tanh(u) &= \operatorname{sech}^2(u)\frac{du}{dx}\,, & \frac{d}{dx}\coth(u) &= -\operatorname{csch}^2(u)\frac{du}{dx}\,,\\
\frac{d}{dx}\operatorname{sech}(u) &= -\operatorname{sech}(u)\tanh(u)\frac{du}{dx}\,, & \frac{d}{dx}\operatorname{csch}(u) &= -\operatorname{csch}(u)\coth(u)\frac{du}{dx}\,.
\end{aligned} \tag{3.22}$$

It is best if you know these formulas too. If you should forget any one of them however, you should be able to recover it as outlined above. Formulas (3.21) and (3.22) are very similar to those for the derivatives of the corresponding trigonometric functions; all that changes is two signs.

**3.6: Derivatives of Inverse Functions.** Because a function $f$ is "undone" when composed with its inverse function $f^{-1}$ in the sense that $u = f(f^{-1}(u))$, the chain rule can be used to derive the **inverse function rule:**

$$\frac{d}{dx}f^{-1}(u) = \frac{1}{f'\big(f^{-1}(u)\big)}\,\frac{du}{dx}\,. \tag{3.23}$$

One could state this as "the derivative of an inverse is the reciprocal of the derivative", provided you realize that this leaves a lot unsaid about the argument of the derivative in the denominator.

One does not need to memorize (3.23) as much as to master its derivation. It goes like this. To find the derivative formula for $v = f^{-1}(u)$, first take the derivative of the identity $f(v) = u$ to obtain

$$f'(v)\,\frac{dv}{dx} = \frac{du}{dx}\,.$$

Then solve for $dv/dx$ and use $v = f^{-1}(u)$ to eliminate the $v$ in $f'(v)$. This gives (3.23). The hardest part of the derivation remains because you can usually use identities to simplify $f'(f^{-1}(u))$.

When the above proceedure is applied to the trigonometric functions, one finds the **inverse trigonometric rules:**

$$\begin{aligned}
\frac{d}{dx}\sin^{-1}(u) &= \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} & \frac{d}{dx}\cos^{-1}(u) &= -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\\
\frac{d}{dx}\tan^{-1}(u) &= \frac{1}{1+u^2}\frac{du}{dx} & \frac{d}{dx}\cot^{-1}(u) &= -\frac{1}{1+u^2}\frac{du}{dx}\\
\frac{d}{dx}\sec^{-1}(u) &= \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} & \frac{d}{dx}\csc^{-1}(u) &= -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}
\end{aligned} \tag{3.24}$$

You should know how to use the to derive these formulas. I would say that you should know those in the left column. Those in the right column have derivatives that differ from those in the left column by only a sign; this is a consequence of the co-function relations for the inverse trigonometric functions.

When the above proceedure is applied to the hyperbolic functions, one finds the **inverse hyperbolic rules:**

$$\begin{aligned}
&\frac{d}{dx}\sinh^{-1}(u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} &\qquad& \frac{d}{dx}\cosh^{-1}(u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx} \\
&\frac{d}{dx}\tanh^{-1}(u) = \frac{1}{1-u^2}\frac{du}{dx} && \frac{d}{dx}\coth^{-1}(u) = -\frac{1}{u^2-1}\frac{du}{dx} \\
&\frac{d}{dx}\operatorname{sech}^{-1}(u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx} && \frac{d}{dx}\operatorname{csch}^{-1}(u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}
\end{aligned} \tag{3.25}$$

These need not be memorized. You should however know how to derive these formulas.

**3.7: Derivatives of Functions Defined Implicitly.** A function $u$ that is defined by the requirement that $y = u(x)$ gives a solution of an equation

$$G(x, y) = 0\,, \tag{3.26}$$

is said to be defined implicitly by (3.26). Solving such an equation for $y$ to obtain $u(x)$ explicitly is generally very difficult, or most likely, impossible. The **method of implicit differentiation** allows you to compute values of $dy/dx$ without obtaining $u(x)$ explicitly. The idea is to imagine that $y$ is given by a known function of $x$ and to just differentiate (3.26) using the various rules for diffentiating combinations given in this section (product rule, chain rule, etc.). The result of this calculation will be something of the form

$$0 = \frac{d}{dx}G(x, y) = A(x, y)\frac{dy}{dx} + B(x, y)\,. \tag{3.27}$$

This can be viewed as a linear equation for $dy/dx$, which yields

$$\frac{dy}{dx} = -\frac{B(x, y)}{A(x, y)} \qquad \text{wherever } A(x, y) \neq 0\,. \tag{3.28}$$

Hence, $dy/dx$ can be evaluated at any $(x, y)$ that satisfies (3.26) for which $A(x, y) \neq 0$.

## 4. DERIVATIVES OF ELEMENTARY FUNCTIONS

Below is a list of derivative formulas for elementary functions in chain rule form.

**4.1: General Power Functions.**

$$\frac{d}{dx}u^p = p\,u^{p-1}\frac{du}{dx}$$

This is just a combination of (3.9) and (3.16).

**4.2: Exponential Functions.**

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} \qquad\qquad \frac{d}{dx}a^u = \ln(a)\,a^u\frac{du}{dx}$$

These are just a restatement of (3.14).

**4.3: Logarithmic Functions.**

$$\frac{d}{dx}\ln(u) = \frac{1}{u}\,\frac{du}{dx} \qquad\qquad \frac{d}{dx}\log_a(u) = \frac{1}{\ln(a)}\,\frac{1}{u}\,\frac{du}{dx}$$

These are just a restatement of (3.15).

**4.4: Trigonometric Functions.**

$$\begin{aligned}
&\frac{d}{dx}\sin(u) = \cos(u)\frac{du}{dx} &\qquad& \frac{d}{dx}\cos(u) = -\sin(u)\frac{du}{dx}\\
&\frac{d}{dx}\tan(u) = \sec^2(u)\frac{du}{dx} && \frac{d}{dx}\cot(u) = -\csc^2(u)\frac{du}{dx}\\
&\frac{d}{dx}\sec(u) = \sec(u)\tan(u)\frac{du}{dx} && \frac{d}{dx}\csc(u) = -\csc(u)\cot(u)\frac{du}{dx}
\end{aligned}$$

This is just a combination of (3.19) and (3.20).

**4.5: Hyperbolic Functions.**

$$\begin{aligned}
&\frac{d}{dx}\sinh(u) = \cosh(u)\frac{du}{dx} &\qquad& \frac{d}{dx}\cosh(u) = \sinh(u)\frac{du}{dx}\\
&\frac{d}{dx}\tanh(u) = \operatorname{sech}^2(u)\frac{du}{dx} && \frac{d}{dx}\coth(u) = -\operatorname{csch}^2(u)\frac{du}{dx}\\
&\frac{d}{dx}\operatorname{sech}(u) = -\operatorname{sech}(u)\tanh(u)\frac{du}{dx} && \frac{d}{dx}\operatorname{csch}(u) = -\operatorname{csch}(u)\coth(u)\frac{du}{dx}
\end{aligned}$$

This is just a combination of (3.21) and (3.22). These formulas are very similar to those for the derivatives of the trigonometric functions; all that changes is two signs.

**4.6: Inverse Trigonometric Functions.**

$$\frac{d}{dx}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}\cos^{-1}(u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}\tan^{-1}(u) = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}\cot^{-1}(u) = -\frac{1}{1+u^2}\frac{du}{dx}$$
$$\frac{d}{dx}\sec^{-1}(u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad \frac{d}{dx}\csc^{-1}(u) = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

These are just a restatement of (3.24). You should know how to derive these formulas. I would say that you should know those in the left column. Those in the right column have derivatives that differ from those in the left column by only a sign; this is a consequence of the co-function relations for the inverse trigonometric functions.

**4.7: Inverse Hyperbolic Functions.**

$$\frac{d}{dx}\sinh^{-1}(u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} \qquad \frac{d}{dx}\cosh^{-1}(u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$
$$\frac{d}{dx}\tanh^{-1}(u) = \frac{1}{1-u^2}\frac{du}{dx} \qquad \frac{d}{dx}\coth^{-1}(u) = -\frac{1}{u^2-1}\frac{du}{dx}$$
$$\frac{d}{dx}\mathrm{sech}^{-1}(u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}\mathrm{csch}^{-1}(u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$$

These are just a restatement of (3.25). These need not be memorized. You should however know how to derive these formulas.