A REVIEW OF TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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The following is a review of commonly used identities involving trigonometric and hyperbolic functions. It is intended to supplement your existing knowledge, as well as the material from the book and class lectures. It is not my intention that you memorize all of the identities contained herein, but rather that you familiarize yourself with them. Of course, the ones identified as very important should be memorized. The others can then be easily recovered by the steps indicated in the text. If you are not already, you should become familiar with these steps to the point were they seem obvious to you. When you have reached that stage you will have mastered, rather than have memorized, the identities. In order to facilitate that process, the identities are presented in conceptually related groups. The many striking parallels between trigonometric and hyperbolic functions should also prove helpful. These skills will serve you in two ways. First, they will minimize the number of formulas that you have to memorize. Second, they will enable you to more easily recover the hundreds of identities that are not listed on these sheets.

1. TRIGONOMETRIC FUNCTIONS

1.1: Definitions. The most important trigonometric functions are $\sin(x)$ and $\cos(x)$. As is discussed in the book, they both are defined for every x and oscillate between -1 and 1. If you understand $\sin(x)$ and $\cos(x)$, you can understand all the other common trigonometric functions through their **definitions:**

$$\begin{aligned} \tan(x) &\equiv \frac{\sin(x)}{\cos(x)}\,, & \cot(x) &\equiv \frac{\cos(x)}{\sin(x)}\,, \\ \sec(x) &\equiv \frac{1}{\cos(x)}\,, & \csc(x) &\equiv \frac{1}{\sin(x)}\,. \end{aligned} \tag{1.1}$$

These are all very important and should be known. Notice that $\tan(x)$ and $\sec(x)$ are not defined where $\cos(x) = 0$, while $\cot(x)$ and $\csc(x)$ are not defined where $\sin(x) = 0$. If you are not so already, use your calculator or a book to become familiar with the graphs of these functions.

1.2: Relationships. It is easily seen from their definitions that these functions are *interrelated* in many ways. For example, one has the **reciprocal relations:**

$$\cot(x) = \frac{1}{\tan(x)}\,, \qquad \sec(x) = \frac{1}{\cos(x)}\,, \qquad \csc(x) = \frac{1}{\sin(x)}\,. \tag{1.2}$$

These either follow directly from or are restatements of (1.1) and should be known. Most calculators have buttons for only $\sin(x)$, $\cos(x)$, and $\tan(x)$, so relations (1.2) are commonly used to generate the other functions. One also has the **co-function relations:**

$$\begin{aligned} \cos(x) &= \sin(\tfrac{\pi}{2} - x)\,, & \sin(x) &= \cos(\tfrac{\pi}{2} - x)\,, \\ \cot(x) &= \tan(\tfrac{\pi}{2} - x)\,, & \tan(x) &= \cot(\tfrac{\pi}{2} - x)\,, \\ \csc(x) &= \sec(\tfrac{\pi}{2} - x)\,, & \sec(x) &= \csc(\tfrac{\pi}{2} - x)\,. \end{aligned} \tag{1.3}$$

The top two of these are very important and should be known. The others then follow easily from definitions (1.1). Notice that the right and left columns can be obtained from each other by simply replacing x by $\frac{\pi}{2} - x$. Finally, most fundamental of all are the **Pythagorean relations:**

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1\,, \\ \sec^2(x) - \tan^2(x) &= 1\,, \\ \csc^2(x) - \cot^2(x) &= 1\,. \end{aligned} \tag{1.4}$$

The first these is very important and should be known. The second and third can be easily recovered from the first by dividing it by $\cos(x)$ and $\sin(x)$ respectively, using definitions (1.1), and making an obvious rearrangement of the terms.

1.3: Symmetries. It is evident from their graphs that each trigonometric function enjoys many *symmetries*—that is, relations to itself. For example, there are the **periodic symmetries:**

$$\begin{aligned} &\sin(x+2\pi)=\sin(x)\,, \qquad &&\cos(x+2\pi)=\cos(x)\,, \qquad &&\tan(x+\pi)=\tan(x)\,, \\ &\csc(x+2\pi)=\csc(x)\,, \qquad &&\sec(x+2\pi)=\sec(x)\,, \qquad &&\cot(x+\pi)=\cot(x)\,. \end{aligned} \tag{1.5}$$

The top three of these are very important and should be known. The bottom three are obtained from the ones immediately above them through the reciprocal relations (1.2). In addition, there are the **anti-periodic symmetries:**

$$\begin{aligned} &\sin(x+\pi)=-\sin(x)\,, \qquad &&\cos(x+\pi)=-\cos(x)\,, \\ &\csc(x+\pi)=-\csc(x)\,, \qquad &&\sec(x+\pi)=-\sec(x)\,. \end{aligned} \tag{1.6}$$

The top two of these are very important and should be known. The bottom two are again obtained from the ones immediately above them through the reciprocal relations (1.2). Finally, there are the **even/odd symmetries:**

$$\begin{aligned} &\sin(-x)=-\sin(x)\,, \qquad &&\cos(-x)=\cos(x)\,, \qquad &&\tan(-x)=-\tan(x)\,, \\ &\csc(-x)=-\csc(x)\,, \qquad &&\sec(-x)=\sec(x)\,, \qquad &&\cot(-x)=-\cot(x)\,. \end{aligned} \tag{1.7}$$

The first two on the top are very important and should be known. The others then follow immediately from definitions (1.1). Can all of the other symmetries that you see by looking at the graphs of these functions be built up from the ones listed in (1.5–1.7)?

1.4: Addition Formulas. The identities that describe how trigonometric functions change when their argument undergoes an arithmetic operation are generally known as *addition formulas*, because addition is the most fundamental arithmetic operation. These identities are often laid out in long lists as if they should all be memorized. In fact, they all derive from a few basic ones in very simple ways. So it is best to just memorize those few, and learn to recover the others. Those few are the three **angle sum formulas:**

$$\begin{aligned} \sin(x+y)&=\sin(x)\cos(y)+\cos(x)\sin(y)\,, \\ \cos(x+y)&=\cos(x)\cos(y)-\sin(x)\sin(y)\,, \\ \tan(x+y)&=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\,. \end{aligned} \tag{1.8}$$

The first two of these are the most important because the third is easily recovered by dividing the first formula by the second. By replacing y by $-y$ in (1.8) and using the even/odd symmetries (1.7), one is led to the **angle difference formulas:**

$$\begin{aligned} \sin(x-y)&=\sin(x)\cos(y)-\cos(x)\sin(y)\,, \\ \cos(x-y)&=\cos(x)\cos(y)+\sin(x)\sin(y)\,, \\ \tan(x-y)&=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}\,. \end{aligned} \tag{1.9}$$

If you set $y = x$ in formulas (1.8) you obtain the **double-angle formulas:**

$$
\begin{aligned}
\sin(2x) &= 2\sin(x)\cos(x)\,, \\
\cos(2x) &= \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)\,, \\
\tan(2x) &= \frac{2\tan(x)}{1-\tan^2(x)}\,.
\end{aligned}
\tag{1.10}
$$

The alternative forms given for $\cos(2x)$ follow from the first of the Pythagorean relations (1.4). The formulas for $\sin(2x)$ and $\cos(2x)$ are so commonly used that they are good to know too. By setting $y = 2x$ in the angle sum formulas, you can continue on to obtain triple-angle formulas and so on, but we will not give them here. The last two formulas for $\cos(2x)$ given in (1.10) lead to the **half-angle formulas:**

$$
\begin{gathered}
\sin(x/2) = \pm\left(\frac{1-\cos(x)}{2}\right)^{\frac{1}{2}}, \qquad \cos(x/2) = \pm\left(\frac{1+\cos(x)}{2}\right)^{\frac{1}{2}}, \\
\tan(x/2) = \pm\left(\frac{1-\cos(x)}{1+\cos(x)}\right)^{\frac{1}{2}} = \frac{\sin(x)}{1+\cos(x)} = \frac{1-\cos(x)}{\sin(x)}\,.
\end{gathered}
\tag{1.11}
$$

The top two derive from the last two formulas for $\cos(2x)$ by simply replacing x by $x/2$ and making an obvious rearrangement of the terms. Care must be taken to pick the correct sign for the square root. The first equality of the third is easily recovered by dividing the first formula by the second. The alternative forms given for $\tan(x/2)$ follow from the first of the Pythagorean identities (1.4). There is no ambiguity of sign in those forms.

Other members of the addition formula family include the **product formulas:**

$$
\begin{aligned}
\cos(x)\cos(y) &= \tfrac{1}{2}\big[\cos(x-y) + \cos(x+y)\big]\,, \\
\sin(x)\sin(y) &= \tfrac{1}{2}\big[\cos(x-y) - \cos(x+y)\big]\,, \\
\sin(x)\cos(y) &= \tfrac{1}{2}\big[\sin(x-y) + \sin(x+y)\big]\,.
\end{aligned}
\tag{1.12}
$$

The first two of these easily follow by adding and subtracting the cosine angle sum and difference formulas. The last easily follows by adding the sine angle sum and difference formulas. The final members of the addition formula family that we will give here are the **sum formulas:**

$$
\begin{aligned}
\cos(x) + \cos(y) &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right), \\
\sin(x) + \sin(y) &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).
\end{aligned}
\tag{1.13}
$$

These can be viewed as being derived from the first and last formulas of (1.12) by replacing x by $(x+y)/2$ and y by $(y-x)/2$. Like the product fromulas, these need not be memorized so long as you can recall the basic idea behind their derivation.

2. INVERSE TRIGONOMETRIC FUNCTIONS

2.1: Definitions. Because they are periodic, trigonometric functions do not have inverses when considered over their whole domain. They do however have inverses if we consider them over restricted domains. We chose the restricted domains as follows:

$$\begin{array}{cc} \underline{\text{function}} & \underline{\text{restricted domain}} \\ y=\sin(x) & -\frac{\pi}{2}\le x\le\frac{\pi}{2} \\ y=\cos(x) & 0\le x\le\pi \\ y=\tan(x) & -\frac{\pi}{2}< x<\frac{\pi}{2} \\ y=\cot(x) & 0< x<\pi \\ y=\sec(x) & 0\le x<\frac{\pi}{2}\text{ and }\frac{\pi}{2}< x\le\pi \\ y=\csc(x) & -\frac{\pi}{2}\le x<0\text{ and }0< x\le\frac{\pi}{2} \end{array} \tag{2.1}$$

The above choices of resticted domains for $\sin(x)$, $\cos(x)$ and $\tan(x)$ are fairly universal, but there are differences from book to book on the choices of resticted domains for $\cot(x)$, $\sec(x)$ and $\csc(x)$. With these choices of restricted domains, the inverse trigonometric functions have the following domains and ranges:

$$\begin{array}{ccc} \underline{\text{inverse function}} & \underline{\text{domain}} & \underline{\text{range}} \\ y=\sin^{-1}(x) & |x|\le 1 & -\frac{\pi}{2}\le y\le\frac{\pi}{2} \\ y=\cos^{-1}(x) & |x|\le 1 & 0\le y\le\pi \\ y=\tan^{-1}(x) & \text{all } x & -\frac{\pi}{2}< y<\frac{\pi}{2} \\ y=\cot^{-1}(x) & \text{all } x & 0< y<\pi \\ y=\sec^{-1}(x) & |x|\ge 1 & 0\le y<\frac{\pi}{2}\text{ and }\frac{\pi}{2}< y\le\pi \\ y=\csc^{-1}(x) & |x|\ge 1 & -\frac{\pi}{2}\le y<0\text{ and }0< y\le\frac{\pi}{2} \end{array} \tag{2.2}$$

Do you understand the various inequalities that give the domains and ranges of these inverse functions? These should be evident by examining their graphs.

2.2: Relationships. One consequence of our choice of restricted domains is that the inverse trigonometric functions satisfy the **co-function relations:**

$$\begin{aligned} \sin^{-1}(x)+\cos^{-1}(x) &= \tfrac{\pi}{2}\,, \\ \tan^{-1}(x)+\cot^{-1}(x) &= \tfrac{\pi}{2}\,, \\ \sec^{-1}(x)+\csc^{-1}(x) &= \tfrac{\pi}{2}\,. \end{aligned} \tag{2.3}$$

These reflect the co-function relations (1.3).

The reciprocal relations (1.2) are reflected in the **reciprocal relations:**

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right), \qquad \csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right), \tag{2.4a}$$

which hold for every x such that $|x| \geq 1$, and

$$\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) & \text{for every } x > 0\,, \\ \tan^{-1}\left(\frac{1}{x}\right) + \pi & \text{for every } x < 0\,. \end{cases} \tag{2.4b}$$

Neither (2.3) nor (2.4) need be memorized, but you should be familiar enough with the inverse trigonometric functions that they can be recovered with a little thought. Can you think of relations between inverse trigonometric functions that reflect the Pythagorean relations (1.4)?

2.3: Symmetries. It is evident from their graphs that the inverse trigonometric functions have less symmetries than the trigonometric functions. For example, they are not periodic or anti-periodic. They do however have the **odd symmetries:**

$$\begin{aligned} \sin^{-1}(-x) &= -\sin^{-1}(x)\,, \\ \tan^{-1}(-x) &= -\tan^{-1}(x)\,, \\ \csc^{-1}(-x) &= -\csc^{-1}(x)\,. \end{aligned} \tag{2.5}$$

These arise because $\sin(x)$, $\tan(x)$ and $\csc(x)$ each have odd symmetry (1.7) *and* because each of the restricted domain used to define their inverses is symmetric under change of sign. Notice that $\cot^{-1}(x)$ is not odd because the restricted domain used to define it is not symmetric under change of sign. When the odd symmetries (2.5) are combined with the co-function relations (2.3), they yield the **shifted odd symmetries:**

$$\begin{aligned} \cos^{-1}(-x) + \cos^{-1}(x) &= \pi\,, \\ \cot^{-1}(-x) + \cot^{-1}(x) &= \pi\,, \\ \sec^{-1}(-x) + \sec^{-1}(x) &= \pi\,. \end{aligned} \tag{2.6}$$

Once again, neither (2.5) nor (2.6) need be memorized, but you should be familiar enough with the inverse trigonometric functions that they can be recovered with a little thought. Can you see these symmetries in the graphs of the inverse trigonometric functions?

3. HYPERBOLIC FUNCTIONS

3.1: Definitions. The most important hyperbolic functions are $\sinh(x)$ and $\cosh(x)$, which are given by the **definitions:**

$$\sinh(x) \equiv \frac{e^x - e^{-x}}{2}\,, \qquad \cosh(x) \equiv \frac{e^x + e^{-x}}{2}\,. \tag{3.1}$$

Both functions are defined for every x. If you understand these functions, you can understand all the other common hyperbolic functions through their **definitions:**

$$\begin{aligned} \tanh(x) &\equiv \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}\,, &\qquad \coth(x) &\equiv \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}\,, \\ \operatorname{sech}(x) &\equiv \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}\,, &\qquad \operatorname{csch}(x) &\equiv \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}\,. \end{aligned} \tag{3.2}$$

These are all very important and should be known. Notice that $\tanh(x)$ and $\operatorname{sech}(x)$ are defined for every x because $\cosh(x) > 0$, while $\coth(x)$ and $\operatorname{csch}(x)$ are not defined where $\sinh(x) = 0$, which only happens at $x = 0$. If you are not so already, use your calculator or a book to become familiar with the graphs of these functions.

3.2: Relationships. It is easily seen from their definitions that these functions are interrelated. For example, one has the **reciprocal relations:**

$$\coth(x) = \frac{1}{\tanh(x)}\,, \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}\,, \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}\,. \tag{3.3}$$

These either follow directly from or are restatements of (3.2) and should be known. Most fundamental of all are the **Pythagorean relations:**

$$\begin{aligned} \cosh^2(x) - \sinh^2(x) &= 1\,, \\ \operatorname{sech}^2(x) + \tanh^2(x) &= 1\,, \\ \coth^2(x) - \operatorname{csch}^2(x) &= 1\,. \end{aligned} \tag{3.4}$$

The first these is very important and should be known. The second and third can be easily recovered from the first by dividing it by $\cosh(x)$ and $\sinh(x)$ respectively, using definitions (3.2), and making an obvious rearrangement of the terms.

3.3: Symmetries. It is evident from their graphs that the hyperbolic functions have the **even/odd symmetries:**

$$\begin{aligned} \sinh(-x) &= -\sinh(x)\,, &\quad \cosh(-x) &= \cosh(x)\,, &\quad \tanh(-x) &= -\tanh(x)\,, \\ \operatorname{csch}(-x) &= -\operatorname{csch}(x)\,, &\quad \operatorname{sech}(-x) &= \operatorname{sech}(x)\,, &\quad \coth(-x) &= -\coth(x)\,. \end{aligned} \tag{3.5}$$

The first two on the top can be seen directly from definitions (3.1), but are very important and should be known. The others then follow immediately from definitions (3.2).

3.4: Addition Formulas. Finally, hyperbolic functions also satisfy a family of addition formulas that describe how they change when their argument undergoes an arithmetic operation. This family is very similar to the analogous family for trigonometric functions. As in that case, the identities all derive from a few basic ones in very simple ways. So it is best to just memorize those few, and learn to recover the others. Those few are the three **argument sum formulas:**

$$\begin{aligned} \sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y)\,, \\ \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y)\,, \\ \tanh(x+y) &= \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}\,. \end{aligned} \tag{3.6}$$

The first two of these are very important and should be known. The third is easily recovered by dividing the first formula by the second. By replacing y by $-y$ in (3.6) and using the even/odd symmetries (3.5), one is led to the **argument difference formulas:**

$$\begin{aligned} \sinh(x-y) &= \sinh(x)\cosh(y) - \cosh(x)\sinh(y)\,, \\ \cosh(x-y) &= \cosh(x)\cosh(y) - \sinh(x)\sinh(y)\,, \\ \tanh(x-y) &= \frac{\tanh(x)-\tanh(y)}{1-\tanh(x)\tanh(y)}\,. \end{aligned} \tag{3.7}$$

By setting $y = x$ in formulas (3.6), you obtain the **double-argument formulas:**

$$\begin{aligned} \sinh(2x) &= 2\sinh(x)\cosh(x)\,, \\ \cosh(2x) &= \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 1 + 2\sinh^2(x)\,, \\ \tanh(2x) &= \frac{2\tanh(x)}{1+\tanh^2(x)}\,. \end{aligned} \tag{3.8}$$

The alternative forms given for $\cosh(2x)$ follow from the first of the Pythagorean identities (3.4). The last two formulas for $\cosh(2x)$ given in (3.8) lead to the **half-argument formulas:**

$$\begin{gathered} \sinh(x/2) = \operatorname{sign}(x)\left(\frac{\cosh(x)-1}{2}\right)^{\frac{1}{2}}, \qquad \cosh(x/2) = \left(\frac{\cosh(x)+1}{2}\right)^{\frac{1}{2}}, \\ \tanh(x/2) = \operatorname{sign}(x)\left(\frac{\cosh(x)-1}{\cosh(x)+1}\right)^{\frac{1}{2}} = \frac{\sinh(x)}{\cosh(x)+1} = \frac{\cosh(x)-1}{\sinh(x)}\,. \end{gathered} \tag{3.9}$$

The top two derive from the last two formulas for $\cosh(2x)$ by simply replacing x by $x/2$ and making an obvious rearrangement of the terms. The first equality of the third is easily recovered by dividing the first formula by the second. The alternative forms given for $\tanh(x/2)$ follow from the first of the Pythagorean identities (3.4).

Other members of the addition formula family include the **product formulas:**

$$\begin{aligned} \cosh(x)\cosh(y) &= \tfrac{1}{2}\big[\cosh(x+y)+\cosh(x-y)\big]\,, \\ \sinh(x)\sinh(y) &= \tfrac{1}{2}\big[\cosh(x+y)-\cosh(x-y)\big]\,, \\ \sinh(x)\cosh(y) &= \tfrac{1}{2}\big[\sinh(x+y)+\sinh(x-y)\big]\,. \end{aligned} \tag{3.10}$$

The first two of these easily follow by adding and subtracting the cosh argument sum and difference formulas. The last easily follows by adding the sinh argument sum and difference formulas. The final members of the addition formula family that we will give are the **sum formulas:**

$$\begin{aligned} \cosh(x)+\cosh(y) &= 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)\,, \\ \sinh(x)+\sinh(y) &= 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)\,. \end{aligned} \tag{3.11}$$

These can be viewed as being derived from the first and last formulas of (3.10) by replacing x by $(x+y)/2$ and y by $(x-y)/2$. Like the product fromulas, these need not be memorized so long as you can recall the basic idea behind their derivation.

4. INVERSE HYPERBOLIC FUNCTIONS

4.1: Definitions. The odd hyperbolic functions ($\sinh(x)$, $\tanh(x)$, $\coth(x)$ and $\operatorname{csch}(x)$) have inverses when considered over their whole domain, while the even ones ($\cosh(x)$ and $\operatorname{sech}(x)$) do not. However, the even ones do have inverses if we consider them over the restricted domain $x \geq 0$. The inverse hyperbolic functions will then have the following domains and ranges:

inverse function	domain	range
$y = \sinh^{-1}(x)$	all x	all y
$y = \cosh^{-1}(x)$	$x \geq 1$	$y \geq 0$
$y = \tanh^{-1}(x)$	$\lvert x\rvert < 1$	all y
$y = \coth^{-1}(x)$	$\lvert x\rvert > 1$	$y \neq 0$
$y = \operatorname{sech}^{-1}(x)$	$0 < x \leq 1$	$y \geq 0$
$y = \operatorname{csch}^{-1}(x)$	$x \neq 0$	$y \neq 0$

(4.1)

Do you understand the various inequalities that give the domains and ranges of these inverse functions? These should be evident by examining their graphs.

Just as the hyperbolic functions have explicit formulas in terms of exponentials, with logarithms one can write down the **inverse formulas:**

$$
\begin{aligned}
\sinh^{-1}(x) &= \ln\left(x + \sqrt{x^2+1}\right) && \text{for every } x\,,\\
\cosh^{-1}(x) &= \ln\left(x + \sqrt{x^2-1}\right) && \text{for every } x \geq 1\,,\\
\tanh^{-1}(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) && \text{for every } |x| < 1\,,\\
\coth^{-1}(x) &= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) && \text{for every } |x| > 1\,,\\
\operatorname{sech}^{-1}(x) &= \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) && \text{for every } 0 < x \leq 1\,,\\
\operatorname{csch}^{-1}(x) &= \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) && \text{for every } x \neq 0\,.
\end{aligned}
\tag{4.2}
$$

These formulas need not be memorized because they may be easily derived.

4.2: Relationships. The reciprocal relations for hyperbolic functions (3.3) are reflected in the **reciprocal relations:**

$$
\begin{aligned}
\coth^{-1}(x) &= \tanh^{-1}\left(\frac{1}{x}\right) && \text{for every } |x| > 1\,,\\
\operatorname{sech}^{-1}(x) &= \cosh^{-1}\left(\frac{1}{x}\right) && \text{for every } 0 < x \leq 1\,,\\
\operatorname{csch}^{-1}(x) &= \sinh^{-1}\left(\frac{1}{x}\right) && \text{for every } x \neq 0\,.
\end{aligned}
\tag{4.3}
$$

These need not be memorized, but you should be familiar enough with inverse hyperbolic functions that they can be recovered with a little thought. Can you think of relations between inverse hyperbolic functions that reflect the Pythagorean relations (3.4)?

4.3: Symmetries. It is evident from their graphs that the inverse hyperbolic functions have the **odd symmetries:**

$$
\begin{aligned}
\sinh^{-1}(-x) &= -\sinh^{-1}(x)\,,\\
\tanh^{-1}(-x) &= -\tanh^{-1}(x)\,,\\
\coth^{-1}(-x) &= -\coth^{-1}(x)\,,\\
\operatorname{csch}^{-1}(-x) &= -\operatorname{csch}^{-1}(x)\,.
\end{aligned}
\tag{4.4}
$$

These arise because $\sinh(x)$, $\tanh(x)$, $\coth(x)$ and $\operatorname{csch}(x)$ each have odd symmetry (3.5) *and* because no restriction of their domains was needed in order to define their inverses.

5. EXERCISES

1. Use the relations (1.3) and/or the symmetries (1.5–1.7) to show:

a. $\sin(\frac{\pi}{2}+x) = \sin(\frac{\pi}{2}-x)$, b. $\cos(\frac{\pi}{2}+x) = -\cos(\frac{\pi}{2}-x)$,
c. $\tan(\frac{\pi}{2}+x) = -\tan(\frac{\pi}{2}-x)$, d. $\sec(\frac{\pi}{2}+x) = -\sec(\frac{\pi}{2}-x)$.

2. Use addition formulas to show:

a. $\sin(3x) = 3\sin(x) - 4\sin^3(x)$,

b. $\sinh(4x) = 4\sinh(x)\cosh(x)[\cosh^2(x) + \sinh^2(x)]$,

c. $\tan(3x) = \dfrac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)}$.

3. Use the reciprocal relations (1.2) and (3.3) to show:

a. $\sec^{-1}(x) = \cos^{-1}(1/x)$ for every $|x| \geq 1$,

b. $\operatorname{csch}^{-1}(x) = \sinh^{-1}(1/x)$ for every $x \neq 0$,

c. relation (2.4b) holds.

4. Use Pythagorean relations to show:

a. $\cos^{-1}(x) = \sin^{-1}\left(\sqrt{1-x^2}\right)$ for every $0 \leq x \leq 1$,

b. $\sec^{-1}(x) = \tan^{-1}\left(\sqrt{x^2-1}\right)$ for every $x \geq 1$,

c. $\sinh^{-1}(x) = \cosh^{-1}\left(\sqrt{1+x^2}\right)$ for every $x \geq 0$.

5. Illustrate the identities a and b of Problem 4 with appropriately drawn right triangles. Hint: In each case one side of the triangle will be of length x and another of length 1.

6. Derive the formula for $\sinh^{-1}(x)$ given in (4.2). Hint: To find $y = \sinh^{-1}(x)$ you must solve $x = \sinh(y)$ for y. But by (3.1) you see that

$$x = \sinh(y) = \frac{e^y - e^{-y}}{2} = \frac{1}{2}\left(u - \frac{1}{u}\right),$$

where $u = e^y$. Solve this equation for u in terms of x and then use ln to find y.

7. Derive the formula for $\coth^{-1}(x)$ given in (4.2). Hint: Follow the strategy suggested by the hint for Problem 6.