MORE BASICS OF INTEGRATION

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This is a survey of some basic facts about integration that will be covered on our final exam. It supplements the material covered in the book (Chapter 6 and Sections 7.1 and 7.2) and in the class lectures. It covers primitives (antiderivatives) viewed analytically and graphically, the Second Fundamental Theorem of Calculus, indefinite integrals, differential equations, uniform motion and acceleration, and substitution applied to both indefinite and definite integrals. Section 3 gives a table of indefinite integrals with elementary forms. Section 5 gives an overview of strategies for evaluating definite integrals.

1. PRIMITIVES AND DEFINITE INTEGRALS

Recall that if f and F are functions such that F is differentiable and

$$\frac{d}{dx}F(x) = f(x)\,,$$

then f is called the derivative of F and F is called a primitive (or an antiderivative) of f. Primitives and definite integrals are connected through the Fundamental Theorem of Calculus.

1.1: General Primitives. Notice that while every differentiable function F has a unique derivative F', primitives of a function f are not unique! Indeed, one has the following facts.

(1) If F is a primitive of f then so is $F + k$ for every constant k. This is because

$$\frac{d}{dx}\big(F(x) + k\big) = \frac{d}{dx}F(x) = f(x)\,.$$

(2) If F and G are primitives of f then there exists a constant k such that $G = F + k$. This is because

$$\frac{d}{dx}\big(G(x) - F(x)\big) = \frac{d}{dx}G(x) - \frac{d}{dx}F(x) = f(x) - f(x) = 0\,,$$

whereby $G - F$ is a constant.

In light of these facts, if F is any primitive of f then we say that the **general primitive** (or the **general antiderivative**) of f is the *family of functions* given by $F + k$ where k is an arbitrary constant.

Fact (2) above tells us that when using the Fundamental Theorem of Calculus to evaluate a definite integral, it does not matter which primitive of the integrand is used. This is because if F and G are primitives of f then by Fact (2) there exists a constant k such that $G = F + k$, whereby

$$G(b) - G(a) = \big(F(b) + k\big) - \big(F(a) + k\big) = F(b) + k - F(a) - k = F(b) - F(a)\,.$$

Hence,

$$\int_a^b f(x)\,dx = G(b) - G(a) = F(b) - F(a)\,. \tag{1.1}$$

Because the constant k drops out of the above calculation, its value does not impact the value obtained for the definite integral.

Facts (1) and (2) tell us that if a function f has a primitive F, it has the whole family of primitives of the form $F+k$. In order to specify a particular member of this family, one additional piece of information must be given. This is most commonly done by specifying the value of the primitive at some point. For example, if you seek the primitive $G = F+k$ with a given value $G(c)$ at some point c, the constant k is uniquely determined by the relation $G(c) = F(c) + k$.

1.2: Constructing Primitives. The above considerations do not tell us whether or not any given function f has a primitive. The Fundamental Theorem of Calculus tells us that if f is integrable and **IF** it has a primitive then the unique primitive F with a given value $F(c)$ at some point c satisfies

$$F(x) = F(c) + \int_c^x f(q)\,dq\,. \tag{1.2}$$

At this point the assumption that f has a primitive is a big one because you have no criteria yet for knowing when this is the case. This section will provide such criteria.

Formula (1.2) does suggest that if f is integrable then we might be able to simply *define* a function G that would be a primitive of f by

$$G(x) = K + \int_c^x f(q)\,dq\,, \tag{1.3}$$

where K is an arbitrary constant. This is not quite the case. Indeed, not every integrable function f has a primitive. What goes wrong with construction (1.3) is that there may be points x where $G'(x) \neq f(x)$. However, at points x where f is continuous one can indeed show that $G'(x) = f(x)$. Hence, if f is continuous over the whole interval then construction (1.3) yields a primitive of f. We state this very important fact as a theorem.

The Second Fundamental Theorem of Calculus. If f is continuous over an interval containing the point c then for any given K, the function G defined by (1.3) is the unique primitive of f with $G(c) = K$.

To prove this theorem one must apply the definition of the derivative to G defined by (1.3) and use the continuity of f at x to show that $G'(x) = f(x)$. One finds that

$$G'(x) = \lim_{h \to 0} \frac{G(x+h) - G(x)}{h}\,,$$

where

$$\frac{G(x+h) - G(x)}{h} = \frac{1}{h}\left(\int_c^{x+h} f(q)\,dq - \int_c^x f(q)\,dq\right) = \frac{1}{h}\int_x^{x+h} f(q)\,dq\,.$$

The last expression gives the average value of f between x and $x+h$. Because f is continuous at x one can argue that

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(q)\, dq = f(x)\,,$$

and thereby conclude that $G'(x) = f(x)$.

Recall that the First Fundamental Theorem of Calculus stated that if f is differentiable and f' is integrable then

$$f(c) + \int_c^x f'(q)\, dq = f(x)\,. \tag{1.4}$$

In contrast, the Second Fundamental Theorem of Calculus states that if f is continuous then

$$\frac{d}{dx} \int_c^x f(q)\, dq = f(x)\,. \tag{1.5}$$

Roughly speaking, the First says that if you integrate the derivative of a function then you get back the function, while the Second says that if you differentiate an integral of a function then you get back the function. Taken together, they state that differentiation and integration are "inverse operations" of each other (at least when they are applied to certain functions) in the sense that (1.4) and (1.5) state they undo each other. This was the key observation that enabled the development of Calculus by Newton and Leibniz, and hence why these theorems have the designation 'fundamental'.

1.3: A Graphical Understanding of Primitives. Graphically, the general primitive of a function f can be understood as the family of vertical shifts of the graph of any one primitive F. In order to specify a particular primitive, one commonly specifies the value of the primitive at some point. Graphically it is clear that any value can be specified at any point where F is defined because that value can be realized by an appropriate vertical shift of F.

Given the graph of f over an interval $[a, b]$, you should be able to easily sketch a graph of any primitive F, say the one with $F(c) = 0$. By the Second Fundamental Theorem of Calculus the value of $F(p)$ at any point p is given by

$$F(p) = \int_c^x f(x)\, dx\,,$$

which you can approximate by estimating the signed area of the region bounded by $x = c$, $x = p$, the x-axis, and the graph of f. Which points p you choose to use should be guided by information that you can read off from the graph of f. For example, the sign of f tells you that:

F is increasing over intervals where f is positive;

F is decreasing over intervals where f is negative;

F is constant over intervals where f vanishes;

F has a local minimum where f changes sign from negative to positive;

F has a local maximum where f changes sign from positive to negative.

In addition, the monotonicity of f tells you that:

F is concave up over intervals where f is increasing;

F is concave down over intervals where f is decreasing;

F is linear over intervals where f is constant;

F has an upward inflection where f has a local minimum;

F has an downward inflection where f has a local maximum.

Notice that all this information can easily be read off without trying to construct f'.

Finally, more information might be read off by considering possible symmetries of f. For example, you can see that:

F is even if f is odd;

F is odd if f is even and $F(0) = 0$;

F has period p if f has period p and an average value of zero;

F has antiperiod p if f has antiperiod p and F has an average value of zero.

1.4: Definite Integrals with Variable Endpoints of Integration. The argument following the statement of the Second Fundamental Theorem of Calculus shows that the derivative of a definite integral with respect to its upper endpoint of integration is just its integrand evaluated at the upper endpoint, provided the integrand is continuous there. That is, it shows that

$$\frac{d}{dx} \int_c^x f(q)\, dq = f(x) \,, \tag{1.6}$$

provided f is continuous at x. By exchanging the endpoints of integration above one can show that

$$\frac{d}{dz} \int_z^c f(q)\, dq = -f(z) \,,$$

provided f is continuous at z. More generally, if the upper and lower endpoints of integration are differentiable functions of a variable t then the derivative of the definite integral with respect to t is given by

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(q)\,dq = f(b(t))\,b'(t) - f(a(t))\,a'(t)\,, \tag{1.7}$$

provided f is continuous. This follows from (1.6), (1.7), and the chain rule because

$$\begin{aligned}\frac{d}{dt}\int_{a(t)}^{b(t)} f(q)\,dq &= \frac{d}{dt}\big(F(b(t)) - F(a(t))\big) = F'(b(t))\,b'(t) - F'(a(t))\,a'(t)\\ &= f(b(t))\,b'(t) - f(a(t))\,a'(t)\,.\end{aligned}$$

The usefulness of (1.7) is that it does not require that you know a primitive of f to compute such a derivative. For example, one has

$$\begin{aligned}&\frac{d}{dt}\int_0^{\sin(t)} e^{z^2}\,dz = e^{\sin^2(t)}\,\cos(t)\,,\\ &\frac{d}{dx}\int_{1+x^2}^{e^x} \cos(t^3)\,dt = \cos(e^{3x})\,e^x - \cos((1+x^2)^3)\,2x\,,\\ &\frac{d}{dz}\int_{\sinh(z)}^{\cosh(z)} \sqrt{v^4+1}\,dv = (\cosh^4(z)+1)^{\frac{1}{2}}\,\sinh(z) - (\sinh^4(z)+1)^{\frac{1}{2}}\,\cosh(z)\,.\end{aligned}$$

2. INDEFINITE INTEGRALS AND DIFFERENTIAL EQUATIONS

2.1: Indefinite Integrals. Let f is a function that is defined over an interval containing a point c and integrable over every bounded subinterval containing c. The family of functions of the form (1.3) is then called the **indefinite integral** of f and is denoted as an integral without endpoints. Hence, one writes

$$\int f(x)\,dx = \int_c^x f(q)\,dq + K\,, \tag{2.1}$$

where K is an arbitrary constant that gives the value of the function at the point c. This family of functions does not depend on the choice of c. (Do you see why?) If f also has a primitive F then the Fundamental Theorem of Calculus shows that

$$\int f(x)\,dx = F(x) + C\,, \tag{2.2}$$

where the constant C is related to K by $C = K - F(c)$. In other words, whenever f has a primitive, the indefinite integral and the general primitive of f are identical.

Unlike a definite integral, which is a number, an indefinite integral is a *family of functions*. It is expressed in terms of the variable of integration and has an arbitrary additive constant. For example, one has

$$\int x^2\,dx = \frac{1}{3}x^3 + C\,, \qquad \int e^t\,dt = e^t + K\,.$$

In these examples the variables of integration are x and t, while the arbitrary constants are C and K respectively.

Like definite integrals, indefinite integrals enjoy the property of having linear dependence on the integrand. Specifically, if f and g are functions and k is a constant then

$$\begin{aligned} \int k\,f(x)\,dx &= k\int f(x)\,dx\,,\\ \int f(x)+g(x)\,dx &= \int f(x)\,dx + \int g(x)\,dx\,. \end{aligned} \tag{2.3}$$

These are also expressed in words as "the integral of a multiple is the multiple of the integral" and "the integral of a sum is the sum of the integrals" respectively.

An indefinite integral is said to have an elementary form if it can be integrated by just applying one of the derivative formulas that you know backwards. Section 3 contains a table of such formulas. Most integrals are not so easy to integrate. Sometimes you can use (2.3) or an identity to change the integrand into ones that have elementary forms. For example, consider

$$\begin{aligned}
\int \left(\frac{w^2+1}{w}\right)^2 dw &= \int \left(w + \frac{1}{w}\right)^2 dw \\
&= \int w^2 + 2 + \frac{1}{w^2}\, dw = \frac{1}{3}w^3 + 2w - \frac{1}{w} + C\,, \\
\int \frac{1}{\cos^2(t)}\, dt &= \int \sec^2(t)\, dt = \tan(t) + C\,, \\
\int \tan^2(z)\, dz &= \int \sec^2(z) - 1\, dz = \tan(z) - z + C\,.
\end{aligned}$$

2.2: Differential Equations. The equation

$$\frac{dy}{dx} = f(x) \tag{2.4}$$

is an example of a simple **differential equation**. In general, differential equations indirectly relate variables by giving a direct relation between their derivatives. For example, equation (2.4) directly relates dy/dx with x through f, and thereby indirectly relates the variables x and y. A solution of this equation is $y = F(x)$ provided F is any primitive of f. The **general solution** of this equation is then $y = F(x) + C$. By (2.2) this general solution could expressed as

$$y = \int f(x)\, dx\,.$$

For example, the differential equation

$$\frac{dy}{dx} = \sin(3x)$$

has as its general solution

$$y = \int \sin(3x)\, dx = -\frac{1}{3}\cos(3x) + C\,.$$

In order to render the solution of a differential equation unique, an additional condition must be imposed. For an equation of the form (2.4) this condition usually takes the form of specifying the value of y at some point x_0. That is to say, one considers the problem

$$\frac{dy}{dx} = f(x)\,, \qquad y = y_0 \text{ at } x = x_0\,. \tag{2.5}$$

The new condition is sometimes called an **initial condition**, while y_0 is called an **initial value**. It is clear that this is enough to specify the arbitrary constant in the general solution. Indeed, given the general solution $y = F(x) + C$, the initial condition states that $y_0 = F(x_0) + C$, whereby $C = y_0 - F(x_0)$. Alternatively, this solution can be expressed as

$$y = F(x) + y_0 - F(x_0) = y_0 + \int_{x_0}^{x} f(q)\, dq\,. \tag{2.6}$$

(Keep in mind that q is a dummy variable here.) Both approaches can be used to solve an initial value problem of the form (2.5).

To illustrate, consider the problem

$$\frac{dy}{dx} = \sin(3x)\,, \qquad y = 1 \text{ at } x = 0\,.$$

The general solution is $y = -\frac{1}{3}\cos(3x) + C$. The initial condition states that $1 = -\frac{1}{3} + C$, whereby $C = \frac{4}{3}$. The unique solution is therefore $y = -\frac{1}{3}\cos(3x) + \frac{4}{3}$. Alternatively, by (2.6) one has

$$y = 1 + \int_0^x \sin(3t)\, dt = 1 - \frac{1}{3}\cos(3t)\bigg|_0^x = 1 - \frac{1}{3}\cos(3x) + \frac{1}{3} = \frac{4}{3} - \frac{1}{3}\cos(3x)\,.$$

You should master both approaches. If you can find the primitive easily, you might find the first one quicker. Otherwise, you might find the second quicker.

2.3: Uniform Motion and Acceleration. An object is said to be in **uniform motion** if it moves with a constant velocity v_0. Its position s satisfies the differential equation

$$\frac{ds}{dt} = v_0\,,$$

which has the general solution $s = v_0 t + s_0$ where s_0 is the initial position.

An object is said to have **uniform acceleration** if its acceleration is a constant a_0. Its velocity v satisfies the differential equation

$$\frac{dv}{dt} = a_0\,,$$

which has the general solution $v = a_0 t + v_0$ where v_0 is now the initial velocity. Its position s satisfies the differential equation

$$\frac{ds}{dt} = v = a_0 t + v_0\,,$$

which has the general solution $s = \frac{1}{2}a_0 t^2 + v_0 t + s_0$ where s_0 is the initial position. This is the situation for objects accelerated by gravity near the surface of the Earth. In that case $a_0 = -g$ where $g \approx 32\,\mathrm{ft/sec}^2 \approx 9.81\,\mathrm{m/sec}^2 = 981\,\mathrm{cm/sec}^2$. The minus sign indicates that gravitational acceleration is downward.

3. INDEFINITE INTEGRALS WITH ELEMENTARY FORMS

The formulas for the derivatives of power functions and the natural logarithm lead to the following formula for indefinite integrals of power functions:

$$\int u^p \, du = \begin{cases} \dfrac{1}{p+1} u^{p+1} + C & \text{for } p \neq -1 \,, \\ \ln(|u|) + C & \text{for } p = -1 \,. \end{cases} \tag{3.1}$$

The formulas for the derivatives of exponential functions lead to the following formulas for indefinite integrals of exponential functions:

$$\int e^u \, du = e^u + C \,, \qquad \int a^u \, du = \frac{1}{\ln(a)} \, a^u + C \,. \tag{3.2}$$

The formulas for the derivatives of the trigonometric and hyperbolic functions lead to the following formulas for indefinite integrals of trigonometric and hyperbolic functions:

$$\begin{aligned}
&\int \cos(u) \, du = \sin(u) + C \,, && \int \cosh(u) \, du = \sinh(u) + C \,, \\
&\int \sin(u) \, du = -\cos(u) + C \,, && \int \sinh(u) \, du = \cosh(u) + C \,, \\
&\int \sec^2(u) \, du = \tan(u) + C \,, && \int \operatorname{sech}^2(u) \, du = \tanh(u) + C \,, \\
&\int \csc^2(u) \, du = -\cot(u) + C \,, && \int \operatorname{csch}^2(u) \, du = -\coth(u) + C \,, \\
&\int \sec(u)\tan(u) \, du = \sec(u) + C \,, && \int \operatorname{sech}(u)\tanh(u) \, du = -\operatorname{sech}(u) + C \,, \\
&\int \csc(u)\cot(u) \, du = -\csc(u) + C \,, && \int \operatorname{csch}(u)\coth(u) \, du = -\operatorname{csch}(u) + C \,.
\end{aligned} \tag{3.3}$$

Notice the similarities between the formulas involving trigonometric functions and those involving hyperbolic functions.

The formulas for the derivatives of the inverse trigonometric and inverse hyperbolic functions lead to many formulas for indefinite integrals of algebraic functions. Of those, the most useful are:

$$\begin{aligned}
&\int \frac{1}{\sqrt{1-u^2}} \, du = \sin^{-1}(u) + C \,, && \int \frac{1}{\sqrt{u^2-1}} \, du = \cosh^{-1}(u) + C \,, \\
&\int \frac{1}{1+u^2} \, du = \tan^{-1}(u) + C \,, && \int \frac{1}{\sqrt{1+u^2}} \, du = \sinh^{-1}(u) + C \,.
\end{aligned} \tag{3.4}$$

4. SUBSTITUTION

Substitution is often called the most powerful technique in all of mathematics. It is certainly the most powerful technique for finding indefinite integrals because it is the technique that undoes the chain rule. As you are certainly aware by now, the chain rule is the most important rule when taking derivatives, so that it should not be surprising that substitution is the most important technique for finding indefinite integrals.

4.1: Substitution Applied to Indefinite Integrals. Suppose you know that F is a primitive of a function f. Then if g is a differentiable function, by the chain rule one has

$$\frac{d}{dx}F(g(x)) = F'(g(x))\,\frac{d}{dx}g(x) = f(g(x))\,g'(x)\,.$$

From this we see that $F(g(x))$ is a primitive of $f(g(x))\,g'(x)$. To 'undo' this, you have to *see* that a given indefinite integral has the form

$$\int f(g(x))\,g'(x)\,dx\,, \tag{4.1}$$

whereby you would know that

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C\,. \tag{4.2}$$

However, to see the form (4.1) for any given integral requires that you have some insight.

This insight can be aided by the technique of substitution. The idea is to choose a g that you think might do the job and introduce a new variable u by $u = g(x)$. This is called the **substitution**. You then compute the **differential** of the substitution by

$$du = \frac{du}{dx}\,dx = g'(x)\,dx\,. \tag{4.3}$$

You then have to *see* if the integral can be expressed in the form

$$\int f(u)\,du\,, \tag{4.4}$$

where you know a primitive F of f. Then because you know that

$$\int f(u)\,du = F(u) + C\,, \tag{4.5}$$

you also know that (4.2) holds. Notice that this is really a technique rather than a method in that it requires you to have the insight to *see* a form (4.4) for which you know (4.5), while it does not give you a recipe for doing so.

Example 4.1: Compute the indefinite integral

$$\int e^{x^2} x \, dx \, .$$

If you let $u = x^2$, so that $du = 2x \, dx$, you *see* that the problem has the form

$$\frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C \, ,$$

whereby you see that

$$\int e^{x^2} x \, dx = \frac{1}{2} e^{x^2} + C \, .$$

Example 4.2: Compute the indefinite integral

$$\int \frac{x^3}{\sqrt{x^4 + 9}} \, dx \, .$$

If you let $u = x^4 + 9$, so that $du = 4x^3 \, dx$, you *see* that the problem has the form

$$\frac{1}{4} \int u^{-\frac{1}{2}} \, du = \frac{1}{2} u^{\frac{1}{2}} + C \, ,$$

whereby you see that

$$\int \frac{x^3}{\sqrt{x^4 + 9}} \, dx = \frac{1}{2} \sqrt{x^4 + 9} + C \, .$$

The technique of substitution has three basic steps:

1. choose $u = g(x)$ and compute the differential $du = g'(x) \, dx$;
2. *see* that with this choice of $g(x)$ the problem has the form

$$\int f(u) \, du = F(u) + C \, ,$$

   where you *know* a primitive F of f;
3. the indefinite integral is then $F(g(x)) + C$.

Step 2 is the hard one. If you do not succeed, go back to Step 1 and try a different $g(x)$! In fact, you should get yourself to the point where you are able to *see* Step 2 rather quickly. To help in this regard, you should become familiar with the elementary forms listed in the last section. They are just the derivative rules that you should already know read backwards. We illustrate some subtleties of Step 2 with more examples.

Example 4.3: Compute the indefinite integral

$$\int \frac{e^t}{(e^t+1)^2}\,dt\,.$$

If you let $u = e^t$, so that $du = e^t\,dt$, the integral has the form

$$\int (u+1)^{-2}\,du\,.$$

While this can be done, its form suggests that a better substitution would have been to let $u = e^t + 1$, so that $du = e^t\,dt$. You then *see* that the problem has the elementary form

$$\int u^{-2}\,du = -u^{-1} + C\,,$$

whereby you see that

$$\int \frac{e^t}{(e^t+1)^2}\,dt = -\frac{1}{e^t+1} + C\,.$$

Example 4.4: Compute the indefinite integral

$$\int \frac{x^7}{\sqrt{x^4+9}}\,dx\,.$$

If you let $u = x^4 + 9$, so that $du = 4x^3\,dx$, you must then *see* x^7 as x^4x^3 with x^3 times dx giving $\frac{1}{4}du$ and x^4 expressed as $x^4 = u - 9$. You thereby *see* that the integral has the form

$$\frac{1}{4}\int (u-9)u^{-\frac{1}{2}}\,du\,,$$

which, while not an elementary form, is a linear combination of two elementary forms. Indeed, you must *see* that

$$\frac{1}{4}\int (u-9)u^{-\frac{1}{2}}\,du = \frac{1}{4}\int (u^{\frac{1}{2}} - 9u^{-\frac{1}{2}})\,du = \frac{1}{6}u^{\frac{3}{2}} - \frac{9}{2}u^{\frac{1}{2}} + C\,,$$

whereby you see that

$$\int \frac{x^7}{\sqrt{x^4+9}}\,dx = \frac{1}{6}(x^4+9)^{\frac{3}{2}} - \frac{9}{2}(x^4+9)^{\frac{1}{2}} + C\,.$$

WARNING: NOT EVERY INDEFINITE INTEGRAL CAN BE DONE IN THIS WAY! Your objective should be to understand the technique well enough to able to identify which can and which can't.

Example 4.5: Compute the indefinite integral

$$\int e^{x^2}\,dx\,.$$

If you let $u = x^2$, so that $du = 2x\,dx$, the integral has the form

$$\frac{1}{2}\int e^u u^{-\frac{1}{2}}\,du\,,$$

which is clearly not an elementary form. No other substitution presents itself.

Example 4.6: Compute the indefinite integral

$$\int \frac{x^2}{\sqrt{x^4+9}}\,dx\,.$$

If you let $u = x^4 + 9$, so that $du = 4x^3\,dx$, the integral has the form

$$\frac{1}{4}\int (u-9)^{-\frac{1}{4}}u^{-\frac{1}{2}}\,du\,,$$

which is clearly neither an elementary form nor a linear combination of elementary forms. On the other hand, if you let $u = x^2$, so that $du = 2x\,dx$, the integral has the form

$$\frac{1}{2}\int u^{\frac{1}{2}}(u^2+9)^{-\frac{1}{2}}\,du\,,$$

which is also clearly neither an elementary form nor a linear combination of elementary forms. No other substitution presents itself.

Remark: Notice the critical role the differential played in each of the above examples. Correctly computing the differential when making a substitution is as fundamental as correctly applying the chain rule when taking a derivative.

Remark: Notice too that in working the above examples, we never mixed x and u in the same integral expression. For instance, in the last example we never wrote

$$\int \frac{x^2}{\sqrt{x^4+9}}\,dx = \frac{1}{2}\int \frac{x}{\sqrt{u^2+9}}\,du\,.$$

There are two reasons for this. First, to do so is sloppy and invites you to commit many errors should you forget that x and u are related by $u = x^2$. Second, and most importantly, it misses the whole point that one should use substitution to *see* the form of the problem. In fact, in all of the above examples, integral expressions involving u appeared only as part of a separate line where the form of the problem was or was not identified. If you work this way, you will find it easier and easier to identify forms in your head, at least up to multiplicative constants.

WARNING: INDEFINITE INTEGRALS THAT CAN BE DONE ONE WAY OFTEN CAN BE DONE MANY WAYS! Different ways of doing an integral may lead to answers that look different at first glance. For example, you may obtain a correct answer that looks different than the answer in the back of the book. Your objective should be to understand integration well enough to see when this is the case.

Example 4.7: Compute the indefinite integral

$$\int \sin(x)\cos(x)\, dx\,.$$

If you let $u = \sin(x)$, so that $du = \cos(x)\, dx$, the integral has the form

$$\int u\, du = \frac{1}{2}u^2 + C\,,$$

whereby

$$\int \sin(x)\cos(x)\, dx = \frac{1}{2}\sin^2(x) + C\,.$$

Alternatively, if you let $u = \cos(x)$, so that $du = -\sin(x)\, dx$, the integral has the form

$$-\int u\, du = -\frac{1}{2}u^2 + C\,,$$

whereby

$$\int \sin(x)\cos(x)\, dx = -\frac{1}{2}\cos^2(x) + C\,.$$

This may seem like a different answer until you remember the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$, and you realize the arbitrary constant C in this answer is not the same as the arbitrary constant in the first answer. At that point you see that

$$-\frac{1}{2}\cos^2(x) + K = -\frac{1}{2}(1-\sin^2(x)) + K = \frac{1}{2}\sin^2(x) + K - \frac{1}{2} = \frac{1}{2}\sin^2(x) + C\,,$$

where the arbitrary constants are related by $C = K - \frac{1}{2}$. One can also do the above indefinite integral by using the double-angle identity $\sin(2x) = 2\sin(x)\cos(x)$. Then

$$\int \sin(x)\cos(x)\, dx = \frac{1}{2}\int \sin(2x)\, dx = -\frac{1}{4}\cos(2x) + C\,.$$

This may also seem like a different answer until you remember the double-angle identity $\cos(2x) = 1 - 2\sin^2(x)$, and you realize the arbitrary constant C in this answer is not the same as the arbitrary constants in the previous answers. At that point you see that

$$-\frac{1}{4}\cos(2x) + K = -\frac{1}{4}(1 - 2\sin^2(x)) + K = \frac{1}{2}\sin^2(x) + K - \frac{1}{4} = \frac{1}{2}\sin^2(x) + C\,,$$

where the arbitrary constants are now related by $C = K - \frac{1}{4}$.

4.2: Substitution Applied to Definite Integrals. Let F be a primitive of a function f and let g be a differentiable function. Then by (4.2) and the Fundamental Theorem of Calculus we know that

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(b)) - F(g(a))\,.$$

On the other hand, the Fundamental Theorem of Calculus also tells us that

$$\int_{g(a)}^{g(b)} f(u)\, du = F(g(b)) - F(g(a))\,.$$

If we combine the above two expressions, we obtain the so-called **change of variable formula** for definite integrals, which holds whether or not you know F:

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du\,. \tag{4.6}$$

This holds whether or not you know F! Notice how the new lower and upper endpoints of integration for u are simply obtained by evaluating $g(x)$ at the lower and upper endpoints of integration for x respectively.

There are therefore two approaches to using substitution to evaluate a definite integral:

I. First use substitution as in the last subsection to find the indefinite integral as a function of the original variable of integration and then apply the endpoints of integration.

II. First use substitution with the change of variables formula to find a definite integral in terms of the new variable of integration and new endpoints of integration. In this approach you never return to the original variable of integration.

You should become comfortable with both approaches. You will find that the second often naturally reduces the arithmetic to be done when evaluating the integral. However, the first may be better if you are not sure of the correct elementary form, and therefore are not sure of the correct substitution.

Example 4.8: Evaluate the definite integral

$$\int_0^2 \frac{x^3}{\sqrt{x^4+9}}\,dx\,.$$

If you use first approach, you let $u = x^4 + 9$, so that $du = 4x^3\,dx$, and you *see* that the indefinite integral has the elementary form

$$\frac{1}{4}\int u^{-\frac{1}{2}}\,du = \frac{1}{2}u^{\frac{1}{2}} + C\,,$$

whereby you see that

$$\int \frac{x^3}{\sqrt{x^4+9}}\,dx = \frac{1}{2}\sqrt{x^4+9} + C\,.$$

Hence, the definite integral can be evaluated as

$$\int_0^2 \frac{x^3}{\sqrt{x^4+9}}\,dx = \frac{1}{2}\sqrt{x^4+9}\,\Big|_0^2 = \frac{1}{2}\big(\sqrt{25}-\sqrt{9}\big) = \frac{1}{2}(5-3) = 1\,.$$

If you use the second approach, you should first *see* that when $u = x^4 + 9$ (so that $du = 4x^3\,dx$) the indefinite integral has the elementary form

$$\frac{1}{4}\int u^{-\frac{1}{2}}\,du = \frac{1}{2}u^{\frac{1}{2}} + C\,.$$

The new endpoints of integration for u are obtained by evaluating $u = x^4 + 9$ at the old endpoints, $x = 0$ and $x = 2$, which yields $u = 0^4 + 9 = 9$ and $u = 2^4 + 9 = 25$. The change of variable formula then gives

$$\int_0^2 \frac{x^3}{\sqrt{x^4+9}}\,dx = \frac{1}{4}\int_9^{25} u^{-\frac{1}{2}}\,du = \frac{1}{2}u^{\frac{1}{2}}\Big|_9^{25} = \frac{1}{2}\big(\sqrt{25}-\sqrt{9}\big) = \frac{1}{2}(5-3) = 1\,.$$

5. EVALUATING DEFINITE INTEGRALS

At this stage, when faced with evaluating a definite integral of the form

$$\int_a^b f(x)\,dx\,,$$

where the function f is given analytically, you have four approaches to consider:

I. symmetry;

II. geometry;

III. the Fundamental Theorem of Calculus;

IV. numerical integration.

Approaches I, II, and III are exact when they apply. Approach IV can always be applied, but is usually approximate. If you go on to study functions of complex variables, you will discover another approach for evaluating certain definite integrals exactly, an incredibly beautiful approach called contour integration. There are others as well. Here however, we restrict our attention to understanding the strengths and weaknesses of the four approaches listed above. This understanding will help you see how to use these approaches either singly or in combination to efficiently and accurately evaluate definite integrals.

5.1: Approach I: Symmetry. This approach can be applied when the integrand and the interval of integration obey a symmetry that allows you to read off the answer (often zero). For example, one sees that

$$\int_{-3}^{3} \sin(x)\tanh(x^4)\,dx = 0\,,$$

because the integrand is odd and the interval of integration is symmetric about the origin. Another example is

$$\int_0^{4\pi} \sinh(\cos(x))\,dx = 0\,,$$

because the integrand has antiperiod π and the integration is over an interval whose length is an even multiple of π. These examples can not be evaluated with either approach II or III. However, even in cases when either approach II or III applies, approach I is quicker if you can apply it. For example,

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin^7(x)\,dx = 0\,,$$

because the integrand is odd and the interval of integration is symmetric about the origin. Another example is

$$\int_{-\pi}^{\pi} \cos^5(x)\, dx = 0\,,$$

because the integrand has antiperiod π and the integration is over an interval whose length is an even multiple of π. The last two integrals can also be done with approach III, but with much more work. (Can you see how?) It is clear that only very few definite integrals can be evaluated with approach I. However, when it applies, the value of the definite integral is obtained very quickly. You should develop an eye for symmetries.

5.2: Approach II: Geometry. This approach can be applied when the integrand describes a geometric region for which you know how to compute the area. For example, one sees that

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}\,,$$

because the region under the curve $y = \sqrt{1-x^2}$ is the upper unit semidisk centered at the origin, the curve being the upper unit semicircle. Another example is

$$\int_{-1}^{1} 1 - |x|\, dx = 1\,,$$

because the region under the curve $y = 1 - |x|$ is a triangle with base 2 and height 1. In the previous two examples the regions described by the integrals were simple geometric shapes for which you know formulas for the area. But the same approach can be used when the region described by the integral can be decomposed into several such simple geometric shapes. For example, you can use this approach to show that

$$\int_{0}^{3} \sqrt{25-x^2}\, dx = \frac{25}{2}\sin^{-1}(3/5) + 6\,,$$

by decomposing the region into a pie slice and a triangle. More generally, you can use a similar decomposition to show for $|b| \le r$ that

$$\int_{0}^{b} \sqrt{r^2-x^2}\, dx = \frac{r^2}{2}\sin^{-1}(b/r) + \frac{b}{2}\sqrt{r^2-b^2}\,.$$

It is clear that only very few definite integrals can be evaluated by this approach. Moreover, such integrals can always be evaluated by approach III. However, when you can apply it, approach II generally gives the answer much quicker. To use it effectively, you must be able to recognize those equations that describe simple geometric shapes.

5.3: Approach III: The Fundamental Theorem of Calculus. This is the most powerful approach. However, it requires that you find a primitive F of the integrand f, after which one simply has

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)\,.$$

The difficulty is in finding the primitive. There is no set method for doing this in general, only a collection of techniques. There are basically three classes of techniques: substitutions, identities, and parts. The first two of these we have already introduced, while the last will be taken up next term.

There are three reasons why you should master these techniques. First, you should know them well enough to be able to identify by inspection those integrands for which primitives can be found, and those techniques by which it can be done. Second, while the task of finding primitives for complicated integrands can be aided by integration tables or by symbolic manipulators like the computer programs Mathematica and Maple, or by calculators like the TI-89 and TI-92, to use such tools effectively sometimes requires that you understand how they get their answers. Third, these techniques extend to more general settings in more advanced courses — settings in which tables and symbolic manipulators are of little use.

5.4: Approach IV: Numerical Integration. This approach can always be used to compute an approximation to any definite integral. Simpson's rule is the most advanced numerical integration method we have covered so far. Calculators actually use methods that are both related to and more advanced than those we covered. However, the methods we covered serve to provide you with an introduction to the world of numerical approximation, which is at the heart of many efforts in science and technology.