Math 246H, First In-Class Exam Solutions

1. (12 points)

   (a) Write a MATLAB command that evaluates the definite integral

   $$\int_0^\infty \frac{r}{1+r^4}\, dr\,.$$

   Solution: The simplest solution is

   int('x/(1+xˆ4)','x',0,inf)

   where you can repace x with any other letter or use Inf instead of inf.

   (b) Sketch the graph that you expect would be produced by the following MATLAB commands.

   [X, Y] = meshgrid(–5:0.2:5,–5:0.2:5)
   contour(X, Y, X.ˆ2 + Y.ˆ2, [25, 25])
   axis square

   Solution: Your sketch should show both the x and y axes marked from -5 to 5 and a single circle of radius 5 centered at the origin. The tick marks on the axes should mark intervals of length .2, but these were optional.

2. (18 points) Give an explicit solution to each of the following initial value problems.

   (a) $\dfrac{dz}{dt} = \dfrac{\cos(t) - z}{1+t}\,, \quad z(0) = 2\,.$

   Solution: This equation is *linear* in z, so write it in the form

   $$\frac{dz}{dt} + \frac{z}{1+t} = \frac{\cos(t)}{1+t}\,.$$

   One sees that an integrating factor μ is given by

   $$\mu = \exp\left(\int \frac{1}{1+t}\, dt\right) = \exp\big(\ln(1+t)\big) = 1+t\,.$$

   Upon multiplying the equation by $\mu = (1+t)$, one finds that

   $$\frac{d}{dt}\big((1+t)z\big) = \cos(t)\,,$$

   which is then integrated to obtain

   $$(1+t)z = \sin(t) + C\,.$$

The value of the integration constant C is found through the initial condition $z(0) = 2$ by setting $t = 0$ and $z = 2$, whereby

$$C = (1 + 0)2 - \sin(0) = 2\,.$$

Hence, upon solving explicitly for z, the solution is

$$z = \frac{2 + \sin(t)}{1 + t}\,.$$

The interval of definition for this solution is $t > -1$.

(b) $\dfrac{du}{dz} = e^u + 1\,, \quad u(0) = 0\,.$

Solution: The equation is *separable*, so write it in the separated differential form as

$$\frac{1}{e^u + 1}\,du = dz\,.$$

This equation can be integrated to obtain

$$z = \int \frac{1}{e^u + 1}\,du = \int \frac{e^{-u}}{1 + e^{-u}}\,du = -\ln(1 + e^{-u}) + C\,.$$

The value of the integration constant C is found through the initial condition $u(0) = 0$ by setting $z = 0$ and $u = 0$, whereby

$$C = 0 + \ln(1 + e^0) = \ln(2)\,.$$

Hence, the solution is given implicitly by

$$z = -\ln(1 + e^{-u}) + \ln(2) = -\ln\left(\frac{1 + e^{-u}}{2}\right).$$

This may be solved explicitly for u as follows:

$$e^{-z} = \frac{1 + e^{-u}}{2}\,;$$

$$2e^{-z} - 1 = e^{-u}\,;$$

$$u = -\ln(2e^{-z} - 1)\,.$$

The interval of definition for this solution is $z < \ln(2)$.

3. (18 points) Consider the differential equation

$$\frac{dy}{dt} = y^3 - 4y\,.$$

(a) Find all of the equilibrium solutions and classify each as stable or unstable.

Solution: Factor the right-hand side as

$$y^3 - 4y = y(y^2 - 4) = y(y+2)(y-2)\,.$$

A sign analysis shows that:

- $\dfrac{dy}{dt} = y(y+2)(y-2) < 0$ when y is in $(-\infty, -2)$ and $(0, 2)$;
- $\dfrac{dy}{dt} = y(y+2)(y-2) > 0$ when y is in $(-2, 0)$ and $(2, \infty)$.

This shows that $y = -2$ and $y = 2$ are unstable equilibrium solutions while $y = 0$ is a stable equilibrium solution.

(b) Draw a graph of y versus t showing the direction field and several solution curves, including all of the equilibrium solutions and solutions above and below each equilibrium value.

Solution:

(c) If $y(0) = 1$, what is the limiting value of y as $t \to \infty$?

Solution: It is clear from the answer to (a) that

- $\dfrac{dy}{dt} < 0$ when y is in $(0, 2)$.

Hence, $y \to 0$ as $t \to \infty$ if $y(0) = 1$.

4. (18 points) A tank initially contains 100 liters of pure water. Beginning at $t = 0$ brine (salt water) with a salt concentration of 2 grams per liter (g/l) flows enters the tank at a constant rate of 3 liters per minute (l/min) and the well-stirred mixture leaves the tank at the same rate. Let $S(t)$ denote the mass of salt in the tank at time $t \geq 0$.

(a) Is $S(t)$ an increasing or decreasing function of t? (Give your reasoning.)

Solution: $S(t)$ is increasing. Because the inflow and outflow rates are equal (both are 3 l/min), $S(t)$ will be increasing as long as the salt concentration of the inflow is greater than that of the outflow, which is the concentration in the tank. This is certainly the case initially, when the tank contains pure water. It will remain the case for all time because as the concentration in the tank increases toward that of the inflow, the rate at which it increases will decrease toward zero.

(b) What is the behavior of $S(t)$ as $t \to \infty$? (Give your reasoning.)

Solution: As $t \to \infty$ the concentration of salt in the tank approaches that of the inflow, which is 2 g/l. Because the tank contains 100 l of brine, this means that $S(t) \to 200$ g as $t \to \infty$.

(c) Derive a formula for $S(t)$.

Solution: The salt concentration in the tank at time t is $S(t)/100$. Because this is also the concentration of the outflow, $S(t)$, the mass of salt in the tank, will satisfy

$$\frac{dS}{dt} = 3 \cdot 2 - 3 \cdot \frac{S}{100}, \qquad S(0) = 0\,.$$

The above differential equation is linear, so we write it as

$$\frac{dS}{dt} + \tfrac{3}{100}\,S = 6\,.$$

One sees that an integrating factor is $e^{\frac{3}{100}t}$, whereby the equation can be recast as

$$\frac{d}{dt}\left(e^{\frac{3}{100}t}S\right) = e^{\frac{3}{100}t}6\,.$$

This is then integrated to obtain

$$e^{\frac{3}{100}t}S = 200e^{\frac{3}{100}t} + C\,.$$

The value of the integration constant C is found through the initial condition $S(0) = 0$ by setting $t = 0$ and $S = 0$, whereby

$$C = e^0 0 - 200e^0 = -200\,.$$

Then solving for S gives

$$S(t) = 200 - 200e^{-\frac{3}{100}t}\,.$$

5. (18 points) Give an implicit general solution to each of the following differential equations.

(a) $\left(\frac{y}{x} + 3x\right)dx + \big(\ln(x) - y\big)dy = 0\,.$

Solution: Because

$$\partial_y\left(\frac{y}{x} + 3x\right) = \frac{1}{x} \quad = \quad \partial_x\big(\ln(x) - y\big) = \frac{1}{x}\,,$$

the equation is *exact*. Hence, we can find $H(x, y)$ such that

$$\partial_x H = \frac{y}{x} + 3x\,, \qquad \partial_y H = \ln(x) - y\,.$$

The first of these shows that

$$H(x,y) = y\ln(x) + \tfrac{3}{2}x^2 + h(y)\,,$$

which with the second equation implies that

$$\ln(x) - y = \partial_y H = \ln(x) + h'(y)\,.$$

Hence, $h'(y) = -y$, or $h(y) = -\frac{1}{2}y^2$. The general solution is therefore given implicitly by

$$y\ln(x) + \tfrac{3}{2}x^2 - \tfrac{1}{2}y^2 = C\,,$$

where C is an arbitrary constant.

(b) $(x^2 + y^3 + 2x)dx + 3y^2 dy = 0\,.$

Solution: Because

$$\partial_y(x^2 + y^3 + 2x) = 3y^2 \quad \neq \quad \partial_x(3y^2) = 0\,,$$

the equation is *not exact*. Seek an integrating factor $\mu(x,y)$ such that

$$\partial_y\big((x^2 + y^3 + 2x)\mu\big) = \partial_x(3y^2\mu)\,.$$

This means that μ must satisfy

$$(x^2 + y^3 + 2x)\partial_y\mu + 3y^2\mu = 3y^2\partial_x\mu\,.$$

If we assume that μ depends only on x (so that $\partial_y\mu = 0$) then this reduces to

$$\mu = \partial_x\mu\,.$$

One sees from this that $\mu = e^x$ is an integrating factor. We can therefore find $H(x,y)$ such that

$$\partial_x H = (x^2 + y^3 + 2x)e^x\,, \qquad \partial_y H = 3y^2 e^x\,.$$

The second of these implies that

$$H(x,y) = y^3 e^x + h(x)\,,$$

which with the first equation implies that

$$(x^2 + y^3 + 2x)e^x = \partial_x H = y^3 e^x + h'(x)\,.$$

Hence, h satisfies

$$h'(x) = (x^2 + 2x)e^x\,.$$

This can be integrated to find $h(x) = x^2 e^x$. The general solution is therefore given implicitly by

$$(y^3 + x^2)e^x = C\,,$$

where C is an arbitrary constant.

6. (16 points) A 2 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance of $v^2/40$ newtons (= kg m/sec^2) where v is the downward velocity of the mass in meters per second. The gravitational acceleration is 9.8 m/sec^2.

   (a) What is the terminal velocity of the mass?

   Solution: The terminal velocity is the velocity at which the force of resistance balances that of gravity. This happens when

$$\tfrac{1}{40}v^2 = mg = 2 \cdot 9.8 \,.$$

   Upon solving for v one obtains

$$v = \sqrt{40 \cdot 2 \cdot 9.8} = 28 \text{ m/sec} \,.$$

   (b) Write down an initial value problem that governs v as a function of time. (You do not have to solve it!)

   Solution: The net downward force on the mass is the force of gravity minus the force of resistance. By Newton ($ma = F$), this leads to

$$m\,\frac{dv}{dt} = mg - \tfrac{1}{40}v^2 \,.$$

   Because $m = 2$ and $g = 9.8$ and the mass is initially at rest, this yields the initial-value problem

$$\frac{dv}{dt} = 9.8 - \tfrac{1}{80}v^2 \,, \qquad v(0) = 0 \,.$$