Math 246H, Second In-Class Exam Solutions

1. (18 points) Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (shown with multiplicities) are $-2+3i$, $-2-3i$, $7i$, $7i$, $-7i$, $-7i$, 5, 5, -3, 0, 0, 0.

   (a) What is the order of L?

   Solution: There are twelve roots listed, so the degree of the characteristic polynomial is twelve, and consequently the order of L must be twelve.

   (b) Give a general real solution of the homogeneous equation $Ly = 0$?

   Solution: The general solution is

$$\begin{aligned} y &= c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) \\ &\quad + c_3\cos(7t) + c_4\sin(7t) + c_5 t\cos(7t) + c_6 t\sin(7t) \\ &\quad + c_7 e^{5t} + c_8 t e^{5t} + c_9 e^{-3t} \\ &\quad + c_{10} + c_{11} t + c_{12} t^2 \,. \end{aligned}$$

   The reasoning is as follows.
   - The conjugate pair $-2 \pm 3i$ yields $e^{-2t}\cos(3t)$ and $e^{-2t}\sin(3t)$.
   - The double conjugate pair $\pm 7i$ yields

$$\cos(7t)\,, \quad \sin(7t)\,, \quad t\cos(7t)\,, \quad \text{and} \quad t\sin(7t)\,.$$

   - The double real root 5 yields e^{5t} and $t\,e^{5t}$.
   - The simple real root -3 yields e^{-3t}.
   - The triple real root 0 yields 1, t, and t^2.

2. (10 points) Suppose you are using the fourth-order Runge-Kutta method to numerically solve an initial-value problem over the interval [0,10]. By what factor would you expect the error to decrease when you increase the number of steps taken from 100 to 300?

   Solution: The global error for the fourth-order Runge-Kutta method is, of course, fourth order. This means that the error scales as h^4, where h is the time step. When you the number of steps taken increases from 100 to 300, the time step h decreases by a factor of 3. The error will therefore decrease like h^4 — namely, by a factor of $3^4 = 81$.

3. (20 points) Solve each of the following initial-value problems.

   (a) $y'' + 4y' + 4y = 0\,, \quad y(0) = 1\,, \quad y'(0) = 0\,.$

   Solution: This is a constant coefficient, homogeneous linear problem. Its characteristic polynomial is

$$P(z) = z^2 + 4z + 4 = (z+2)^2\,.$$

   It has the double real root -2, which yields the general solution

$$y(t) = c_1 e^{-2t} + c_2 t\,e^{-2t}\,.$$

   Because

$$y'(t) = -2c_1 e^{-2t} + c_2\left(e^{-2t} - 2t\,e^{-2t}\right),$$

   when the initial conditions are imposed, one finds that

$$y(0) = c_1 = 1\,, \qquad y'(0) = -2c_1 + c_2 = 0\,.$$

These are solved to find that $c_1 = 1$ and $c_2 = 2$. The solution of the initial-value problem is therefore

$$y(t) = e^{-2t} + 2t\,e^{-2t} = (1 + 2t)\,e^{-2t}\,.$$

(b) $y'' + y = 4e^t\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$

Solution: This is a constant coefficient, inhomogeneous linear problem. The characteristic polynomial of its homogeneous part is

$$P(z) = z^2 + 1\,.$$

It has the conjugate pair of roots $\pm i$, which yields the general homogeneous solution

$$y_H(t) = c_1 \cos(t) + c_2 \sin(t)\,.$$

Because the forcing is of the form e^{zt} for $z = 1$, and because $z = 1$ is not a root of the characteristic polynomial, a particular solution can be found quickly by either the method of undetermined coefficients (as in the book) or the method of determined coefficients (as in class).

The method of *undetermined coefficients* seeks a particular solution of the form $y_P(t) = Ae^t$. Because $y_P''(t) = Ae^t$, one sees that

$$Ly_P = y_P'' + y_P = 2Ae^t = 4e^t\,,$$

which implies $A = 2$. Hence, $y_P(t) = 2e^t$.

The method of *determined coefficients* evaluates the identity $Le^{zt} = P(z)e^{zt}$ at $z = 1$ to obtain $Le^t = 2e^t$. Multiplying this by 2 gives $L(2e^t) = 4e^t$, which shows that $y_P(t) = 2e^t$.

By either method you find $y_P(t) = 2e^t$, and the general solution of the problem is therefore

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + 2e^t\,.$$

Because

$$y'(t) = -c_1 \sin(t) + c_2 \cos(t) + 2e^t\,,$$

when the initial conditions are imposed, one finds that

$$y(0) = c_1 + 2 = 0\,, \qquad y'(0) = c_2 + 2 = 0\,.$$

These are solved to find that $c_1 = -2$ and $c_2 = -2$. The solution of the initial-value problem is therefore

$$y(t) = -2\cos(t) + -2\sin(t) + 2e^t\,.$$

4. (20 points) Give a general solution to each of the following.

(a) $y'' + 4y' + 5y = 3\cos(2t)\,.$

Solution: This is a constant coefficient, inhomogeneous linear problem. The characteristic polynomial of its homogeneous part is

$$P(z) = z^2 + 4z + 5 = (z + 2)^2 + 1\,.$$

It has the conjugate pair of roots $-2 \pm i$, which yields the general homogeneous solution

$$y_H(t) = c_1 e^{-2t} \cos(t) + c_2 e^{-2t} \sin(t)\,.$$

Because the forcing is of the form e^{zt} for $z = i2$, and because $z = i2$ is not a root of the characteristic polynomial, a particular solution can be found quickly by either the method of undetermined coefficients (as in the book) or the method of determined coefficients (as in class).

The method of *undetermined coefficients* seeks a particular solution of the form $y_P(t) = A\cos(2t) + B\sin(2t)$. Because

$$\begin{aligned} y_P'(t) &= -2A\sin(2t) + 2B\cos(2t)\,, \\ y_P''(t) &= -4A\cos(2t) - 4B\sin(2t)\,, \end{aligned}$$

one sees that

$$\begin{aligned} Ly_P &= y_P'' + 4y_P' + 5y_P \\ &= (A + 8B)\cos(2t) + (B - 8A)\sin(2t) = 3\cos(2t)\,. \end{aligned}$$

This leads to the algebraic linear system

$$A + 8B = 3\,, \qquad B - 8A = 0\,.$$

This can be solved to find that $A = 3/65$ and $B = 24/65$. Hence,

$$y_P(t) = \frac{3}{65}\cos(2t) + \frac{24}{65}\sin(2t)\,.$$

The method of *determined coefficients* evaluates the identity $Le^{zt} = P(z)e^{zt}$ at $z = i2$ to obtain $Le^{i2t} = (1 + i8)e^{i2t}$. Multiplying this by $3/(1 + i8)$ shows that

$$L\left(\frac{3}{1 + i8}e^{i2t}\right) = 3e^{i2t}\,.$$

Because $3e^{i2t} = 3\cos(2t) + i3\sin(2t)$, the real part of the left-hand side above will be Ly_P. Because

$$\begin{aligned} \frac{3}{1 + i8}e^{i2t} &= \frac{3(1 - i8)}{65}\big(\cos(2t) + i\sin(2t)\big) \\ &= \left(\frac{3}{65}\cos(2t) + \frac{24}{65}\sin(2t)\right) \\ &\quad + i\left(-\frac{24}{65}\cos(2t) + \frac{3}{65}\sin(2t)\right)\,, \end{aligned}$$

this real part shows that

$$y_P(t) = \frac{3}{65}\cos(2t) + \frac{24}{65}\sin(2t)\,.$$

By either method you find the same y_P, and the general solution of the problem is therefore

$$y = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t) + \frac{3}{65}\cos(2t) + \frac{24}{65}\sin(2t)\,.$$

(b) $y'' - y = e^t\,.$

Solution: This is a constant coefficient, inhomogeneous linear problem. The characteristic polynomial of its homogeneous part is

$$P(z) = z^2 - 1\,.$$

It the simple real roots -1 and 1, which yields the general homogeneous solution

$$y_H(t) = c_1 e^{-t} + c_2 e^t \,.$$

Because the forcing is of the form e^{zt} for $z = 1$, and because $z = 1$ is a root of the characteristic polynomial, a particular solution can be found quickly by either the method of undetermined coefficients (as in the book) or the method of determined coefficients (as in class).

The method of *undetermined coefficients* seeks a particular solution of the form $y_P(t) = Ate^t$. Because

$$y_P'(t) = A(e^t + te^t)\,, \qquad y_P''(t) = A(2e^t + te^t)\,,$$

one sees that

$$Ly_P = y_P'' - y_P = A2e^t = e^t\,,$$

which implies $A = 1/2$. Hence, $y_P(t) = \frac{1}{2}te^t$.

The method of *determined coefficients* evaluates the identity $L(te^{zt}) = P'(z)e^{zt} + P(z)te^{zt}$ at $z = 1$ to obtain $L(te^t) = 2e^t$. Dividing this by 2 gives $L(\frac{1}{2}te^t) = e^t$, which shows that $y_P(t) = \frac{1}{2}te^t$.

By either method you find the same y_P, and the general solution of the problem is therefore

$$y(t) = c_1 e^{-t} + c_2 e^t + \frac{1}{2}te^t\,.$$

5. (16 points) Given that x and x^2 are linearly independent solutions of the homogeneous equation corresponding to

$$x^2y'' - 2xy' + 2y = xe^x\,,$$

find a general solution. You may express the solution in terms of integrals.

Solution: The general solution of this inhomogeneous equation will have the form $y = y_H + y_P$ where y_H is the general solution of the corresponding homogeneous equation and y_P is any particular solution of the inhomogeneous equation. Because you are given that x and x^2 are linearly independent solutions of the corresponding homogeneous equation, you know that

$$y_H = c_1x + c_2x^2\,.$$

The methods of undetermined or determined coefficients cannot be used to find a particular solution, so we will use the method of variation of constants. We first put the equation into its normal form

$$y'' - \frac{2}{x}\,y' + \frac{2}{x^2}\,y = \frac{1}{x}\,e^x\,,$$

and then seek y_P of the form

$$y_P = xu_1(x) + x^2u_2(x)\,.$$

One chooses u_1' and u_2' so that they satisfy

$$xu_1' + x^2u_2' = 0\,, \qquad u_1' + 2xu_2' = \frac{1}{x}\,e^x\,.$$

This linear system is solved to find that

$$u_1' = -\frac{1}{x}\,e^x\,, \qquad u_2' = \frac{1}{x^2}\,e^x\,.$$

Upon integrating these, your answer can be expressed either in terms of indefinite integrals as

$$y = -x \int \frac{1}{x}\, e^x \, dx + x^2 \int \frac{1}{x^2}\, e^x \, dx \,,$$

or in terms of definite integrals as

$$y = c_1 x + c_2 x^2 - x \int_1^x \frac{1}{z}\, e^z \, dz + x^2 \int_1^x \frac{1}{z^2}\, e^z \, dz \,.$$

6. (16 points) Consider the following MATLAB function M-file.
function [t,y] = solveit(ti, yi, tf, n)

```
h = (tf – ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for i = 1:n
z = t(i)^2 + y(i)^2;
t(i + 1) = t(i) + h;
y(i + 1) = y(i) + (h/2)*(z + t(i + 1)^2 + (y(i) + h*z)^2);
end
```

(a) What is the initial-value problem being solved numerically?

Solution: It is solving the initial-value problem

$$\frac{dy}{dt} = t^2 + y^2 \,, \qquad y(t_i) = y_i \quad \text{over } [t_i, t_f] \,.$$

(b) What is the numerical method being used to solve it?

Solution: It is using the improved Euler method.

(c) What are the output values of t(2) and y(2) that you would expect for input values of ti = 1, yi = 1, tf = 5, n = 20 for this method?

Solution: Because it takes 20 timesteps over the inteval $[1, 5]$, the time step h will be given by

$$h = \frac{5-1}{20} = \frac{4}{20} = \frac{1}{5} = .2 \,.$$

Because $t(1) = 1$ and $h = .2$, one sees that $t(2)$ will be given by

$$t(2) = t(1) + h = 1 + .2 = 1.2 \,.$$

Because $y(1) = 1$, $t(2) = 1.2$, $h = .2$, and $z = t(1)^2 + y(1)^2 = 1^2 + 1^2 = 2$, one sees that $y(2)$ will be given by

$$\begin{aligned} y(2) &= y(1) + \frac{h}{2}\left(z + t(2)^2 + (y(1) + h * z)^2\right) \\ &= 1 + \frac{.2}{2}\left(2 + (1.2)^2 + (1 + .2 * 2)^2\right) \\ &= 1 + .1\,(2 + 1.44 + 1.96) = 1.54 \,. \end{aligned}$$