Math 246H, Third In-Class Exam Solutions

1. (12 points) Consider the matrices

$$A = \begin{pmatrix} 3 & 1-i \\ 2+i & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix}.$$

Compute the matrices

(a) A^T,

Solution: The transpose of A is given by

$$A^T = \begin{pmatrix} 3 & 2+i \\ 1-i & 4 \end{pmatrix}.$$

(b) $\overline{A}$,

Solution: The conjugate of A is given by

$$\overline{A} = \begin{pmatrix} 3 & 1+i \\ 2-i & 4 \end{pmatrix}.$$

(c) A^*,

Solution: The adjoint of A is given by

$$A^* = \begin{pmatrix} 3 & 2-i \\ 1+i & 4 \end{pmatrix}.$$

(d) $2A - B$,

Solution: The difference of $2A$ and B is given by

$$2A - B = \begin{pmatrix} 6 & 2-i2 \\ 4+i2 & 8 \end{pmatrix} - \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 0 & -5-i2 \\ -1+i2 & 2 \end{pmatrix}.$$

(e) AB,

Solution: The product of A and B is given by

$$\begin{aligned} AB &= \begin{pmatrix} 3 & 1-i \\ 2+i & 4 \end{pmatrix} \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 3\cdot 6 + (1-i)\cdot 5 & 3\cdot 7 + (1-i)\cdot 6 \\ (2+i)\cdot 6 + 4\cdot 5 & (2+i)\cdot 7 + 4\cdot 6 \end{pmatrix} = \begin{pmatrix} 23-i5 & 27-i6 \\ 32+i6 & 38+i7 \end{pmatrix}. \end{aligned}$$

(f) B^{-1}.

Solution: Observe that it is clear that B has an inverse because

$$\det(B) = \det \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix} = 6\cdot 6 - 5\cdot 7 = 36 - 35 = 1 \neq 0.$$

The inverse of B is given by

$$B^{-1} = \begin{pmatrix} 6 & -7 \\ -5 & 6 \end{pmatrix}.$$

This may be computed in a number of ways. Here are three.

First, it can be computed by applying elementary row operations to transform the augmented matrix $(B \,|\, I)$ into $(I \,|\, B^{-1})$ as follows:

$$\begin{aligned}
\begin{pmatrix} B & \big| & I \end{pmatrix} &= \left(\begin{array}{cc|cc} 6 & 7 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & \frac{7}{6} & \frac{1}{6} & 0 \\ 5 & 6 & 0 & 1 \end{array}\right) \\
&\sim \left(\begin{array}{cc|cc} 1 & \frac{7}{6} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & -\frac{5}{6} & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & \frac{7}{6} & \frac{1}{6} & 0 \\ 0 & 1 & -5 & 6 \end{array}\right) \\
&\sim \left(\begin{array}{cc|cc} 1 & 0 & 6 & -7 \\ 0 & 1 & -5 & 6 \end{array}\right) = \begin{pmatrix} I & \big| & B^{-1} \end{pmatrix} .
\end{aligned}$$

Second, because B is a two-by-two matrix, its inverse can be computed directly from the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \text{whenever } ad - bc \neq 0 .$$

This formula is just formula (24) on page 349 of the book specialized to the two-by-two case. When applied to B it yields

$$B^{-1} = \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix}^{-1} = \frac{1}{36 - 35} \begin{pmatrix} 6 & -7 \\ -5 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ -5 & 6 \end{pmatrix} .$$

Finally, the inverse of B may be computed directly from its definition by seeking a, b, c, and d such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 6a + 7c & 6b + 7d \\ 5a + 6c & 5b + 7d \end{pmatrix} .$$

Then a and c are found by solving the two-by-two system

$$6a + 7c = 1 , \qquad 5a + 6c = 0 ,$$

which gives $a = 6$ and $c = -5$. Similarly b and d are found by solving the two-by-two system

$$6b + 7d = 0 , \qquad 5b + 6d = 1 ,$$

which gives $b = -7$ and $d = 6$. You thereby find that

$$B^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ -5 & 6 \end{pmatrix} .$$

2. (9 points) The vertical displacement of a mass on a spring is given by
$$y(t) = \sqrt{3}\cos(2t) + \sin(2t)\,.$$
Express this in the form $y(t) = A\cos(\omega t - \delta)$, identifying the amplitude and phase of the oscillation.

Solution: The displacement takes the form
$$y(t) = 2\cos\left(2t - \tfrac{\pi}{6}\right),$$
where the amplitude is 2 and the phase is $\frac{\pi}{6}$. There are several approaches to this problem. Here are two.

One approach that requires no memorization other than the usual addition formula for cosine is as follows. Because
$$A\cos(\omega t - \delta) = A\cos(\delta)\cos(\omega t) + A\sin(\delta)\sin(\omega t)\,,$$
this form will be equal to $y(t)$ provided $\omega = 2$ and
$$A\cos(\delta) = \sqrt{3}\,, \qquad A\sin(\delta) = 1\,.$$
Upon solving these equations one finds that the amplitude A is given by
$$A = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2\,,$$
while the phase δ is given by
$$\delta = \sin^{-1}\left(\frac{1}{A}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}\,.$$

Another approach requires you to memorize special formulas for both the amplitude and phase of functions of the form
$$c_1\cos(\omega t) + c_2\sin(\omega t)\,.$$
The formula for the amplitude is easier one because c_1 and c_2 appear in it symmetrically. It gives
$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2\,.$$
The formula for the phase is trickier because c_1 and c_2 do not appear in it symmetrically. It gives
$$\delta = \tan^{-1}\left(\frac{c_2}{c_1}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}\,.$$
The most common mistake made by those who chose this approach was to exchange the roles of c_1 and c_2 in this formula. One way to keep these roles straight is to remember the formula verbally as
$$\text{phase} = \tan^{-1}\left(\frac{\text{coefficient of sine}}{\text{coefficient of cosine}}\right).$$

3. (8 points) Compute the Laplace transform of $f(t) = te^{3t}$ from its definition.

Solution: Let $F(s) = \mathcal{L}\{f\}$. By the definition of the Laplace transform

$$F(s) = \lim_{b\to\infty} \int_0^b e^{-st} te^{3t}\, dt = \lim_{b\to\infty} \int_0^b te^{(3-s)t}\, dt\,.$$

An integration by parts shows that

$$\begin{aligned}\int_0^b t\, e^{(3-s)t}\, dt &= t\, \frac{e^{(3-s)t}}{3-s}\bigg|_0^b - \int_0^b \frac{e^{(3-s)t}}{3-s}\, dt \\ &= \left(t\, \frac{e^{(3-s)t}}{3-s} - \frac{e^{(3-s)t}}{(3-s)^2} \right)\bigg|_0^b \\ &= \left(b\, \frac{e^{(3-s)b}}{3-s} - \frac{e^{(3-s)b}}{(3-s)^2} \right) + \frac{1}{(3-s)^2}\,.\end{aligned}$$

Hence, provided $s > 3$ one has that

$$\begin{aligned}F(s) &= \lim_{b\to\infty} \left[\left(b\, \frac{e^{(3-s)b}}{3-s} - \frac{e^{(3-s)b}}{(3-s)^2} \right) + \frac{1}{(3-s)^2} \right] \\ &= \frac{1}{(3-s)^2} + \lim_{b\to\infty} \left(b\, \frac{e^{-(s-3)b}}{3-s} - \frac{e^{-(s-3)b}}{(3-s)^2} \right) \\ &= \frac{1}{(3-s)^2}\,.\end{aligned}$$

4. (12 points) Consider the matrix

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} .$$

(a) Find all the eigenvalues of A.

Solution: The eigenvalues are the roots of the equation

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{pmatrix} = (5-\lambda)^2 - 3^2 .$$

The eigenvalues are therefore $\lambda = 5 \pm 3$, or simply 2 and 8.

(b) For each eigenvalue of A find an eigenvector.

Solution: A vector x is an eigenvector of A corresponding to an eigenvalue λ provided it is a nonzero solution of $(A - \lambda I)x = 0$.
An eigenvector of A corresponding to $\lambda = 2$ satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5-2 & -3 \\ -3 & 5-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

This leads to the equation $x_1 = x_2$. An eigenvector of A corresponding to the eigenvalue 2 is thereby

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

or any nonzero multiple of it.
An eigenvector of A corresponding to $\lambda = 8$ satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5-8 & -3 \\ -3 & 5-8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

This leads to the equation $x_1 = -x_2$. An eigenvector of A corresponding to the eigenvalue 8 is thereby

$$x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} ,$$

or any nonzero multiple of it.

5. (12 points) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) At $t = 0$ the mass is displaced 3 cm above its equilibrium position and released with no initial velocity. It moves in a medium that imparts a drag force of 2 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. There are no other forces. (As usual, assume the spring force is proportional to displacement and the drag force is proportional to velocity.)

(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve the initial-value problem, just write it down!)

Solution: Let y be the displacement of the mass from the equilibrium position in centimeters, with upward displacements being positive. The governing initial-value problem then has the form

$$my'' + \gamma y' + ky = 0\,, \qquad y(0) = 3\,, \quad y'(0) = 0\,,$$

where m is the mass, γ is the drag coefficient, and k is the spring constant. The problem says that $m = 4$ grams. The spring constant is obtained by balancing the weight of the mass ($mg = 4 \cdot 980$ dynes) with the force applied by the spring when it is stretched 9.8 cm. This gives $k9.8 = 4{\cdot}980$, or

$$k = \frac{4 \cdot 980}{9.8} = 400 \quad \text{g/sec}^2\,.$$

The drag coefficient is obtained by balancing the force of 2 dynes with the drag force imparted by the medium when the speed of the mass is 4 cm/sec. This gives $\gamma 4 = 2$, or

$$\gamma = \frac{2}{4} = \frac{1}{2} \quad \text{g/sec}\,.$$

The governing initial-value problem is therefore

$$4y'' + \frac{1}{2}y' + 400y = 0\,, \qquad y(0) = 3\,, \quad y'(0) = 0\,.$$

If you had chosen downward displacements to be positive then the governing initial-value problem would be identical except for the first initial condition, which would then be $y(0) = -3$.

(b) What is the natural frequency of the spring?

Solution: The natural frequency of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \cdot 980}{9.8 \cdot 4}} = \sqrt{100} = 10 \quad 1/\text{sec}\,.$$

(c) Is the system over damped, critically damped, or under damped? Why?

Solution: The characteristic polynomial is

$$P(z) = z^2 + \frac{1}{8}z + 100 = \left(z + \frac{1}{16}\right)^2 + 100 - \frac{1}{16^2}\,,$$

which has complex roots. The system is therefore under damped.

6. (10 points) Consider the linear algebraic system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1\,, \\ 2x_1 + x_2 + x_3 &= 1\,, \\ x_1 - x_2 + 2x_3 &= 1\,. \end{aligned}$$

Either find its general solution or else show that it has no solution.

Solution: First, a remark. By any method you can compute

$$\det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} = 0\,.$$

This does not mean however that there is no solution. All it means is that there *might* be no solution. Whether there is a solution or not depends on the right-hand side of the system. For example, if the numbers on the right-hand side are 1, 2, and 1 rather than 1, 1, and 1, then the system would have the solution $x_1 = 1$, $x_2 = 0$ and $x_3 = 0$.

In fact, the given system has no solution. There are many ways to see this. One of the simplest is to add the first and third equations to obtain

$$2x_1 + x_2 + x_3 = 2\,,$$

which clearly contradicts the second equation. Done.

You can find similar contradictions by taking other combinations of the equations. For example, subtracting the first from the second contradicts the third. Done.

Alternatively, you can try to solve the system by applying elementary row operations to the augmented matrix. This gives

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & -3 & 3 & 0 \end{array}\right).$$

The last two rows on the right-hand side are clearly contradictory, whereby the system has no solution. Done.

If you did not see the contradiction above and continued to apply elementary row operations, you would get

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & -3 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{3} \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{array}\right).$$

At this point the last row clearly shows that the system has no solution. You have arrived at the right answer, but it took you too long to get there!

7. (12 points) What answer will be produced by the following MatLab commands?

   >> ode1 = 'D2y + 2*Dy + 5*y = 16*exp(t)';
   >> dsolve(ode1, 't')
   ans =

   Solution: These commands ask MatLab to give the general solution of the equation
   $$y'' + 2y' + 5y = 16e^t\,.$$
   MatLab produces the answer

   2*exp(t)+C1*exp(–t)*sin(2*t)+C2*exp(–t)*cos(2*t)

   This can be seen as follows. This is a constant coefficient, inhomogeneous linear equation. The characteristic polynomial of its homogeneous part is
   $$P(z) = z^2 + 2z + 5 = (z+1)^2 + 4\,.$$
   Its roots are $z = -1 \pm i2$, whereby the general solution of the homogeneous part is
   $$y_H(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)\,.$$
   Because the forcing is of the form e^{zt} for $z = 1$, and because $z = 1$ is not a root of the characteristic polynomial, a particular solution can be found quickly by either the method of undetermined coefficients (as in the book) or the method of determined coefficients (as in class).

   The method of *undetermined coefficients* seeks a particular solution of the form $y_P(t) = Ae^t$. Because $y_P'(t) = y_P''(t) = Ae^t$, one sees that
   $$Ly_P = y_P'' + 2y_P' + 5y_P = 8Ae^t = 16e^t\,,$$
   which implies $A = 2$. Hence, $y_P(t) = 2e^t$.

   The method of *determined coefficients* evaluates the identity $Le^{zt} = P(z)e^{zt}$ at $z = 1$ to obtain $Le^t = 8e^t$. Multiplying this by 2 gives $L(2e^t) = 16e^t$, which shows that $y_P(t) = 2e^t$.

   By either method you find $y_P(t) = 2e^t$, and the general solution of the problem is therefore
   $$y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + 2e^t\,.$$
   Up to notational differences, this is the answer that MatLab produces.

8. (10 points) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y' + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \le t < 2\pi\,, \\ 0 & \text{for } t \ge 2\pi\,. \end{cases}$$

You may refer to the table on the last page. (DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$.)

Solution: The Laplace transform of the initial-value problem is

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 13\mathcal{L}\{y\} = \mathcal{L}\{f\}\,,$$

where

$$\begin{aligned} \mathcal{L}\{y\} &= Y(s)\,, \\ \mathcal{L}\{y'\} &= sY(s) - y(0) = sY(s) - 4\,, \\ \mathcal{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s4 - 1\,. \end{aligned}$$

To compute $\mathcal{L}\{f\}$, first rewrite f as

$$f(t) = \big(1 - u(t-2\pi)\big)\cos(t) = \cos(t) - u(t-2\pi)\cos(t-2\pi)\,.$$

Referring to the table on the last page, Item 2 with $b = 1$ and Item 5 with $c = 2\pi$ then show that

$$\begin{aligned} \mathcal{L}\{f\} &= \mathcal{L}\{\cos(t)\} - \mathcal{L}\{u(t-2\pi)\cos(t-2\pi)\} \\ &= \frac{s}{s^2+1} - e^{-2\pi s}\frac{s}{s^2+1} \\ &= \big(1 - e^{-2\pi s}\big)\,\frac{s}{s^2+1}\,. \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 4s - 1\big) + 4\big(sY(s) - 4\big) + 13Y(s) = \big(1 - e^{-2\pi s}\big)\,\frac{s}{s^2+1}\,,$$

which becomes

$$(s^2 + 4s + 13)Y(s) - (4s + 1 + 16) = \big(1 - e^{-2\pi s}\big)\,\frac{s}{s^2+1}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 4s + 13}\left(4s + 17 + \big(1 - e^{-2\pi s}\big)\,\frac{s}{s^2+1}\right).$$

9. (15 points) Find the inverse Laplace tramsform of the following functions. You may refer to the table on the last page.

(a) $F(s) = \dfrac{2}{(s+5)^3}$,

Solution: Referring to the table on the last page, Item 1 with $n = 2$ gives $\mathcal{L}\{t^2\} = 2/s^3$. Item 4 with $a = -5$ and $f(t) = t^2$ then gives

$$\mathcal{L}\{e^{-5t}t^2\} = \frac{2}{(s+5)^3}\,.$$

One therefore finds that

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+5)^3}\right\} = e^{-5t}t^2\,.$$

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}$,

Solution: The denominator factors as $(s-3)(s+2)$ so the partial fraction decomposition is

$$F(s) = \frac{3s}{s^2 - s - 6} = \frac{3s}{(s-3)(s+2)} = \frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}\,.$$

Referring to the table on the last page, Item 1 with $n = 0$ gives $\mathcal{L}\{1\} = 1/s$. Item 4 with $a = 3$, $a = -2$ and $f(t) = 1$ then gives

$$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}\,, \qquad \mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}\,.$$

One therefore finds that

$$\mathcal{L}^{-1}\left\{\frac{3s}{s^2 - s - 6}\right\} = \tfrac{9}{5}e^{3t} + \tfrac{6}{5}e^{-2t}\,.$$

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}$.

Solution: Completion of the square in the denominator gives $(s-2)^2+1$. Referring to the table on the last page, Item 2 with $b = 1$ gives

$$\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}\,.$$

Item 4 with $a = 2$ and $f(t) = \cos(t)$ then gives

$$\mathcal{L}\{e^{2t}\cos(t)\} = \frac{s-2}{(s-2)^2+1}\,.$$

Item 5 with $c = 3$ and $f(t) = e^{2t}\cos(t)$ then gives

$$\mathcal{L}\{u(t-3)e^{2(t-3)}\cos(t-3)\} = e^{-3s}\frac{s-2}{(s-2)^2+1}\,.$$

One therefore finds that

$$\mathcal{L}^{-1}\left\{\frac{(s-2)e^{-3s}}{s^2 - 4s + 5}\right\} = u(t-3)e^{2(t-3)}\cos(t-3)\,.$$

A Short Table of Laplace Transforms

1.	$\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$	for $s > 0$.
2.	$\mathcal{L}\{\cos(bt)\} = \dfrac{s}{s^2 + b^2}$	for $s > 0$.
3.	$\mathcal{L}\{\sin(bt)\} = \dfrac{b}{s^2 + b^2}$	for $s > 0$.
4.	$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$	where $F(s) = \mathcal{L}\{f(t)\}$.
5.	$\mathcal{L}\{u(t - c)f(t - c)\} = e^{-cs} F(s)$	where $F(s) = \mathcal{L}\{f(t)\}$ and u is the step function.