Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 30 January 2018

(1) [4] For each of the following ordinary differential equations, determine its order and whether it is linear or nonlinear. If it is nonlinear, circle a term that makes it so.

(a) $w'''' + \sin(2t)w' = 2w + 5e^t$ **Solution.** fourth order, linear.

(b) $h''' - 2\cos(h)h'' = 4\sin(3t)$ **Solution.** third order, nonlinear, $-2\cos(h)h''$.

(2) [4] Solve the initial-value problem

$$x\frac{\mathrm{d}v}{\mathrm{d}x} + 3v = 5x^2\,, \qquad v(2) = 3\,.$$

Solution. This is a nonhomogeneous linear equation. Its normal form is

$$\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{3}{x}\,v = 5x\,.$$

The coefficient is undefined at $x = 0$ and is continuous elsewhere. The forcing is continuous everywhere. Because the initial point is $x = 2$, the interval of definition will be $(0, \infty)$.

An integrating factor is $e^{A(x)}$ where $A'(x) = 3/x$. Setting $A(x) = 3\log(x)$ for $x > 0$, we obtain $e^{A(x)} = e^{3\log(x)} = x^3$. Hence, the problem has the intgrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x^3 v\big) = x^3 \cdot (5x) = 5x^4\,.$$

Integrating both sides yields

$$x^3 v = x^5 + c\,.$$

Imposing the initial condition gives

$$2^3 \cdot 3 = 2^5 + c\,,$$

whereby $c = 8 \cdot 3 - 32 = 24 - 32 = -8$. Therefore the solution is

$$v = \frac{x^5 - 8}{x^3} = x^2 - \frac{8}{x^3}\,.$$

Group Work Questions for Problem 2 [3]

- Solve the initial-value problem

$$x\frac{\mathrm{d}v}{\mathrm{d}x} + 3v = 5x^2\,, \qquad v(1) = -2\,.$$

- Solve the initial-value problem

$$x\frac{\mathrm{d}v}{\mathrm{d}x} + 3v = 5x^2\,, \qquad v(-1) = 2\,.$$

(3) [2] What is the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{e^z}{z^2 - 64}\, u = \frac{\cos(z)}{z^2 - 16}\,, \qquad u(-6) = 2\,.$$

(You do not have to solve the differential equation to answer this question!)

Solution. This is a nonhomogeneous linear equation that is already in normal form. The coefficient $e^z/(z^2-64)$ is undefined at $z = \pm 8$ and is continuous elsewhere. The forcing $\cos(z)/(z^2-16)$ is undefined at $z = \pm 4$ and is continuous elsewhere. The initial time is $z = -6$. This can be pictured on the z-axis as follows.

$$\begin{array}{ccccccccccc} --\circ-- & \bullet & --\circ-- & + & --+-- & + & --\circ-- & + & --\circ- & \rightarrow & z \\ -8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 & & \end{array}$$

Therefore the interval of definition for the solution is $(-8, -4)$ because:

- the initial time $z = -6$ is in $(-8, -4)$,
- the coefficient and forcing are both continuous over $(-8, -4)$,
- the coefficient is undefined at $z = -8$,
- the forcing is undefined at $z = -4$.

Group Work Questions for Problem 3 [3]

- What is the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{e^z}{z^2 - 64}\, u = \frac{\cos(z)}{z^2 - 16}\,, \qquad u(0) = 3\,.$$

- What is the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{e^z}{z^2 - 64}\, u = \frac{\cos(z)}{z^2 - 16}\,, \qquad u(12) = -1\,.$$

Group Work Preview Questions for Quiz 2 [4]

Consider the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{(u+5)^2(u+1)^3(7-u)}{u-3}\,.$$

Let $u_1(t)$ and $u_2(t)$ be the solutions of it that satisfy $u_1(2) = -3$ and $u_2(-1) = 5$. (You do not have to find these solutions!)

- Sketch the phase-line portrait for the equation.
- Classify each stationary point as being either stable, unstable, or semistable.
- Evaluate $\lim_{t\to\infty} u_1(t)$.
- Evaluate $\lim_{t\to\infty} u_2(t)$.
- Evaluate $\lim_{t\to\infty} \big(u_2(t) - u_1(t)\big)$.