Quiz 2 Solutions, Math 246, Professor David Levermore
Tuesday, 6 February 2018

(1) [5] Find the explicit solution of the initial-value problem

$$\frac{\mathrm{d}w}{\mathrm{d}z} = 3z^2 e^{-w}, \qquad w(-1) = w_I .$$

Give its interval of definition in terms of w_I.

Solution. This differential equation is separable. It is defined everywhere. It has no stationary points. Its separated differential form is $e^w \,\mathrm{d}w = 3z^2 \,\mathrm{d}z$, whereby

$$\int e^w \,\mathrm{d}w = \int 3z^2 \,\mathrm{d}z .$$

Upon integrating both sides we find that

$$e^w = z^3 + c .$$

The initial condition then implies that $e^{w_I} = (-1)^3 + c$, whereby $c = e^{w_I} + 1$. Therefore the solution satisfies

$$e^w = z^3 + e^{w_I} + 1 .$$

We need $z^3 + e^{w_I} + 1 > 0$ for this equation to have a real solution for w, in which case the solution is given by

$$w = \log\big(z^3 + e^{w_I} + 1\big) .$$

The condition $z^3 + e^{w_I} + 1 > 0$ is equivalent to $z^3 > -(e^{w_I} + 1)$, which is equivalent to

$$z > -\big(e^{w_I} + 1\big)^{\frac{1}{3}} .$$

Therefore the interval of definition is

$$\Big(- \big(e^{w_I} + 1\big)^{\frac{1}{3}}, \infty\Big) .$$

Group Work Questions for Problem 1 [3]

- Is the solution an increasing or decreasing function of z?
- How does the solution behave near the left endpoint of this interval?
- How does the solution behaves as $z \to \infty$?
- What is a sketch of the solution?
- What changes if the initial condition is $w(1) = w_I$?

Remark. A plot of this explicit solution can be made for several choices of initial value w_I if you wanted more quantitative information about the solutions. However, a qualitative sketch already gives enough information to address many questions!

(2) [5] Sketch the phase-line portrait for the equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{(v+3)(v-1)(v-3)^2}{(v+1)^2}\,.$$

Classify each stationary point as being either stable, unstable, or semistable. (You do not have to find the solution!)

Solution. This equation is autonomous. Its right-hand side is undefined at $v = -1$ and is differentiable elsewhere. Its stationary points are found by setting

$$\frac{(v+3)(v-1)(v-3)^2}{(v+1)^2} = 0\,.$$

Therefore the stationary points are $v = -3$, $v = 1$, and $v = 3$. A sign analysis of the right-hand side shows that the phase-line portrait is

$$\begin{array}{ccccccccc}
+ & & - & & - & & + & & + \\
\rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \circ & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow \;\; v \\
 & -3 & & -1 & & 1 & & 3 & \\
 & \text{stable} & & \text{undefined} & & \text{unstable} & & \text{semistable} &
\end{array}$$

Remark. Here the terms stable, unstable, and semistable are applied to solutions. The point $v = -1$ is not a solution, so these terms should not be applied to it.

Group Work Questions for Problem 2 [3]

- For what initial values $v(0)$ will $v(t) \to -3$ as $t \to \infty$?
- For what initial values $v(0)$ will $v(t) \to 1$ as $t \to \infty$?
- For what initial values $v(0)$ will $v(t) \to 3$ as $t \to \infty$?
- For what initial values $v(0)$ will $v(t)$ blow up in finite time as t increases?
- For what initial values $v(0)$ will $v'(t)$ blow up in finite time as t increases, but $v(t)$ will not blow up?
- If $v(0)$ lies within the interval $[1, 3]$ then what is the interval of definition of the solution $v(t)$?

Remark. An implicit general solution can be found analytically, but it takes a lot of work. An explicit general solution cannot be found analytically.

Group Work Preview Questions for Quiz 3 [4]

A tank with a capacity of 55 liters initially contains 19 liters of brine (salt water) with a salt concentration of 4 grams per liter (gr/lit). At time $t = 0$ brine with a salt concentration of 7 grams per liter (gr/lit) begins to flow into the tank at a constant rate of 3 liters per minute (lit/min) and the well-stirred mixture flows out of the tank at a constant rate of 1 liters per minute (lit/min).

- Find $V(t)$, the volume of brine in the tank as a function of time for $t > 0$.
- Give the time at which the tank overflows.
- Write down an initial-value problem that governs $S(t)$, the amount of salt in the tank for $t > 0$ until the tank overflows. (Do not solve the initial-value problem!)
- Give the interval of definition for the solution of the initial-value problem.