Quiz 3 Solutions, Math 246, Professor David Levermore
Tuesday, 13 February 2018

(1) [4] In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population such that it would double every seven weeks. There are 240,000 mosquitoes in the area when a flock of birds arrives that eats 30,000 mosquitoes per week. Write down an initial-value problem that governs the population of mosquitoes in the area after the flock of birds arrives. (Do not solve the initial-value problem!)

Solution. Let $M(t)$ be the number of mosquitoes at time t weeks. Doubling every seven weeks gives a growth rate r that satisfies $e^{r7} = 2$, whereby $r = \frac{1}{7}\log(2)$. Therefore the initial-value problem that M satisfies is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \tfrac{1}{7}\log(2)M - 30,000\,, \qquad M(0) = 240,000\,.$$

Questions for Discussion on Problem 1

- Will $M(t)$ be an increasing or decreasing function of t?
- Is the flock of birds large enough to control the mosquitoes?
- How do the answers to the previous two questions change if there were 300,000 mosquitoes in the area when the same flock of birds arrives?

Remark. In the above discussion it is helpful to know that $\frac{2}{3} < \log(2) < \frac{7}{10}$.

(2) [2] Suppose we have used the Runge-trapezoidal method to approximate the solution of an initial-value problem over the time interval $[5, 15]$ with 1000 uniform time steps. About how many uniform time steps do we need to reduce the global error of our approximation by a factor of $\frac{1}{81}$?

Solution. The Runge-trapezoidal method is second order, so its error scales like h^2. To reduce the error by a factor of $\frac{1}{81}$, we must reduce h by a factor of $(\frac{1}{81})^{\frac{1}{2}} = \frac{1}{9}$. We must increase the number of time steps by a factor of 9, which means that we need $9,000$ uniform time steps.

Questions for Discussion on Problem 2

- How does the answer change if the explicit Euler method is being used?
- How does the answer change if the Runge-midpoint method is being used?
- How does the answer change if the Runge-Kutta method is being used?
- How do all the above answers change if our goal was to reduce the global error of our approximation by a factor of $\frac{1}{625}$?

(3) [4] Consider the initial-value problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 3v - v^2 \,, \qquad v(0) = 2 \,.$$

Use one step of the Runge-trapezoidal method to approximate $v(.2)$. Your answer can be left as an arithmetic expression.

Solution. Let $f(v) = 3v - v^2$. Set $v_0 = v(0) = 2$. Then one step of the Runge-trapezoidal method with $h = .2$ yields

$$\begin{aligned}
f_0 &= f(v_0) = 3v_0 - v_0^2 && \text{evaluate } f(v) \text{ at initial value} \\
&\qquad\qquad = 3 \cdot 2 - 2^2 = 6 - 4 = 2 \,, \\
\tilde{v}_1 &= v_0 + hf_0 && \text{full step by explicit Euler} \\
&= 2 + .2 \cdot 2 = 2 + .4 = 2.4 \,, \\
\tilde{f}_1 &= f(\tilde{v}_1) = 3\tilde{v}_1 - \tilde{v}_1^2 && \text{approximate } f(v) \text{ at full step} \\
&\qquad\qquad = 3 \cdot 2.4 - (2.4)^2 \,, \\
v_1 &= v_0 + \frac{h}{2}\big[f_0 + \tilde{f}_1\big] && \text{full step by Runge-trapezoidal} \\
&= 2 + .1\big[2 + \big(3 \cdot 2.4 - (2.4)^2\big)\big] \,.
\end{aligned}$$

Therefore $v(.2) \approx v_1 = 2 + .1\big[2 + 3 \cdot 2.4 - (2.4)^2\big]$.

Remark. You do not have to evaluate the above arithemic expression for full credit. It evaluates to $v(.2) \approx v_1 = 2.344$.

Questions for Discussion on Problem 3

- How does the answer change if two steps of the explicit Euler method are used?
- How does the answer change if one step of the Runge-midpoint method is used?
- The differential equation is autonomous. Use a phase-line portrait to describe how the solution $v(t)$ of the initial-value problem behaves.
  - Is $v(t)$ an increasing or decreasing function of t?
  - How does $v(t)$ behave as $t \to \infty$?

  Are the numerical approximations consistent with this information?
- Find the explicit solution $v(t)$ of the initial-value problem analytically. Compute the exact value of $v(.2)$. How does this compare with the approximations to it found by the numerical methods?