Quiz 4 Solutions, Math 246, Professor David Levermore
Thursday, 22 February 2018

(1) [3] Determine the interval of definition for the solution to the initial-value problem

$$v''' + \frac{1}{2-t}\,v'' - \frac{e^t}{3+t}\,v = \frac{\cos(3t)}{7-t}\,, \qquad v(-1) = v'(-1) = v''(-1) = 5\,.$$

Solution. The nonhomogeneous linear equation for v is already in normal form. Notice that

◇ the coefficient of v'' is undefined at $t = 2$ and is continuous elsewhere;
◇ the coefficient of v is undefined at $t = -3$ and is continuous elsewhere;
◇ the forcing is undefined at $t = 7$ for and is continuous elsewhere;
◇ the initial time is $t = -1$.

Plotting these points on a time-line (not to be confused with a phase-line) gives

$$\text{———}\circ\text{———}\bullet\text{———————}\circ\text{———————————}\circ\text{———}\!\longrightarrow t$$
$$-3 \qquad -1 \qquad\qquad 2 \qquad\qquad\qquad\qquad 7$$

Therefore the interval of definition is $(-3, 2)$ because

- the initial time -1 is in $(-3, 2)$,
- all the coefficients and the forcing are continuous over $(-3, 2)$,
- the coefficient of v is undefined at $t = -3$,
- the coefficient of v'' is undefined at $t = 2$.

Remark. All four reasons must be given for full credit.

∘ The first two reasons are why a (unique) solution exists over the interval $(-3, 2)$.
∘ The last two reasons are why this solution does not exist over a larger interval.

Group Work Questions for Problem 1 [4]

(a) How does the interval of definition change if the initial conditions are

$$v(5) = 0\,, \quad v'(5) = -1\,, \quad v''(5) = 7\,?$$

(b) How does the interval of definition change if the initial conditions are

$$v(-5) = 3\,, \quad v'(-5) = 4\,, \quad v''(-5) = -2\,?$$

(c) Suppose that $V_1(t)$, $V_2(t)$, and $V_3(t)$ are solutions to the differential equation over $(-3, 2)$. Suppose that $\mathrm{Wr}[V_1, V_2, V_3](-1) = 6$. What is $\mathrm{Wr}[V_1, V_2, V_3](t)$ for every t in $(-3, 2)$? Hint: Abel.

(d) Do you expect to be able to find analytic solutions to this initial-value problem?

(e) Suppose that we plan to approximate the solution of the initial-value problem numerically.

  (i) Does it make sense or not to do this over the time interval $[-3.1, 2]$?
  (ii) Does it make sense or not to do this over the time interval $[-3, 2]$?
  (iii) Does it make sense or not to do this over the time interval $[-2.9, 2]$?
  (iv) Does it make sense or not to do this over the time interval $[-2.9, 1.9]$?

(2) [4] Given the fact that e^t and $t\,e^t$ are solutions of $u'' - 2u' + u = 0$, solve the general initial-value problem associated with $t = 0$ — namely, solve

$$u'' - 2u' + u = 0\,, \qquad u(0) = u_0\,, \quad u'(0) = u_1\,.$$

Solution. This is a homogeneous linear equation with constant coefficients. Because we are given that e^t and $t\,e^t$ are solutions to it, we can use the method of linear superposition to seek the solution of the general initial-value problem in the form

$$u(t) = c_1 e^t + c_2 t\,e^t\,.$$

Then $u'(t) = -c_1 e^t + c_2(e^t + t\,e^t)$ and the initial conditions yield

$$u_0 = u(0) = c_1\,, \qquad\qquad u_1 = u'(0) = c_1 + c_2\,.$$

It follows that

$$c_1 = u_0\,, \qquad c_2 = u_1 - u_0\,.$$

Therefore the solution of the general initial-value problem is

$$u = u_0 e^t + (u_1 - u_0)t\,e^t\,.$$

(3) [3] Compute the Wronskian $\mathrm{Wr}[U_1, U_2](t)$ of the functions $U_1(t) = e^t$ and $U_2(t) = t\,e^t$. (Evaluate the determinant and simplify.)

Solution. Because $U_1'(t) = e^t$ and $U_2'(t) = e^t + t\,e^t$, the Wronskian is

$$\begin{aligned}\mathrm{Wr}[U_1, U_2](t) &= \det\begin{pmatrix} U_1(t) & U_2(t) \\ U_1'(t) & U_2'(t)\end{pmatrix} = \det\begin{pmatrix} e^t & t\,e^t \\ e^t & e^t + t\,e^t\end{pmatrix} \\ &= e^t(e^t + t\,e^t)2e^{2t} - e^t(t\,e^t) = e^{2t} + t\,e^{2t} - t\,e^{2t} = e^{2t}\,.\end{aligned}$$

Group Work Questions for Problems 2 and 3 [4]

(a) Why do e^t and $t\,e^t$ comprise a fundamental set of solutions to this equation?
(b) Give a general solution to this equation.
(c) Show that e^t and $t\,e^t$ are linearly independent functions.
(d) How can we know that $\mathrm{Wr}[U_1, U_2](t)$ is proportional to e^{2t} without computing $\mathrm{Wr}[U_1, U_2](t)$? Hint: Abel.
(e) Find the natural fundamental set of solutions for this equation associated with the initial time $t = 0$.
(f) Let $N_0(t)$ and $N_1(t)$ be the natural fundamental set of solutions for this equation associated with the initial time $t = 0$. Show that $N_0(t - t_I)$ and $N_1(t - t_I)$ are the natural fundamental set of solutions for this equation associated with $t = t_I$.

Group Work Preview Questions for Quiz 5 [2]

(1) Suppose that $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$ are solutions of the differential equation

$$y''' + 3y'' + b(t)y' + c(t)y = 0\,,$$

where $b(t)$ and $c(t)$ are continuous, real-valued functions over the time interval $(-7, 5)$. Suppose that $\mathrm{Wr}[Y_1, Y_2, Y_3](0) = 4$. What is $\mathrm{Wr}[Y_1, Y_2, Y_3](-3)$?

(2) Give a real general solution of the equation

$$x'' - 4x' + 3x = 0\,.$$