Quiz 5 Solutions, Math 246, Professor David Levermore
Thursday, 1 March 2018

(1) [4] Given that e^{4t} and e^{-4t} are linearly independent solutions to $x'' - 16x = 0$, find the natural fundamental set of solutions to this equation associated with $t = 0$.

Solution. The general initial-value problem associated with $t = 0$ is

$$x'' - 16x = 0\,, \qquad x(0) = x_0\,, \quad x'(0) = x_1\,.$$

Because e^{4t} and e^{-4t} are linearly independent solutions, we see that a general solution is $x(t) = c_1 e^{4t} + c_2 e^{-4t}$. Then $x'(t) = 4c_1 e^{4t} - 4c_2 e^{-4t}$ and the initial conditions yield

$$\begin{aligned} x_0 &= x(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2\,, \\ x_1 &= x'(0) = 4c_1 e^0 - 4c_2 e^0 = 4c_1 - 4c_2\,. \end{aligned}$$

It follows that $c_1 = \frac{4x_0 + x_1}{8}$ and $c_2 = \frac{4x_0 - x_1}{8}$. Hence, the solution of the general initial-value problem is

$$x(t) = \frac{4x_0 + x_1}{8}\, e^{4t} + \frac{4x_0 - x_1}{8}\, e^{-4t} = x_0 \frac{e^{4t} + e^{-4t}}{2} + x_1 \frac{e^{4t} - e^{-4t}}{8}\,.$$

Therefore the natural fundamental set of solutions associated with $t = 0$ is

$$N_0(t) = \frac{e^{4t} + e^{-4t}}{2}\,, \qquad N_1(t) = \frac{e^{4t} - e^{-4t}}{8}\,.$$

Group Work Questions for Problem 1 [3]

(a) Give the characteristic polynomial associated with this equation. Find its roots. What do they tell you?
(b) Show that e^{4t} and e^{-4t} are linearly independent.
(c) Find $\mathrm{Wr}[N_0, N_1](t)$. Hint: Compute $\mathrm{Wr}[N_0, N_1](0)$ and use Abel.

(2) [5] Give a real general solution to the equation

$$(\mathrm{D}^2 + 4\mathrm{D} + 20)^2 (\mathrm{D} + 3)^4 v = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is an eighth-order, homogeneous, linear differential equation for $v(t)$ with constant coefficients. Its characteristic polynomial is

$$p(\zeta) = (\zeta^2 + 4\zeta + 20)^2 (\zeta + 3)^4 = \big((\zeta + 2)^2 + 4^2\big)^2 (\zeta + 3)^4\,,$$

which has the eight roots

$$-2 + i4\,, \quad -2 + i4\,, \qquad -2 - i4\,, \quad -2 - i4\,, \qquad -3\,, \quad -3\,, \quad -3\,, \quad -3\,.$$

A fundamental set of real-valued solutions is built up as follows.

- ◇ The double conjugate pair of roots $-2 \pm i4$ yields the four real-valued solutions

$$e^{-2t}\cos(4t)\,, \qquad e^{-2t}\sin(4t)\,, \qquad t\,e^{-2t}\cos(4t)\,, \qquad t\,e^{-2t}\sin(4t)\,.$$

- ◇ The quadruple real root -3 yields the four real-valued solutions

$$e^{-3t}\,, \qquad t\,e^{-3t}\,, \qquad t^2 e^{-3t}\,, \qquad t^3 e^{-3t}\,.$$

Therefore a real general solution to the differential equation is

$$v(t) = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t) + c_3 t\, e^{-2t}\cos(4t) + c_4 t\, e^{-2t}\sin(4t)$$
$$+ c_5 e^{-3t} + c_6 t\, e^{-3t} + c_7 t^2 e^{-3t} + c_8 t^3 e^{-3t}\,,$$

where c_1, c_2, c_3, c_4, c_5, c_6, c_7, and c_8 are eight real parameters.

(3) [1] Give a real general solution to the equation

$$u'' + 4u = 15e^t\,,$$

given that $3e^t$ is a solution to it and that $\cos(2t)$ and $\sin(2t)$ are a fundamental set of solutions to the associated homogeneous equation.

Solution. A real general solution to the equation is $u(t) = u_H(t) + u_P(t)$ where $u_H(t)$ is the real general solution to the associated homogeneous equation given by

$$u_H(t) = c_1\cos(2t) + c_2\sin(2t)\,,$$

and $u_P(t)$ is the particular solution to the equation given by $u_P(t) = 3e^t$. Therefore a real general solution to the equation is

$$u(t) = c_1\cos(2t) + c_2\sin(2t) + 3e^t\,.$$

Group Work Questions for Problem 3 [4]

(a) Find the solution to the equation that satisfies $u(0) = 0\,,\quad u'(0) = 1$.
(b) Give the characteristic polynomial associated with this equation. Find its roots. What do they tell you?
(c) Give the degree d, characteristic $\mu + i\nu$, and multiplicity m of the forcing of this equation.
(d) Give the Key Identity associated with this equation and evaluate it at $\mu + i\nu$. What does this tell you?

Group Work Preview Questions for Quiz 6 [3]

(1) Give a real general solution to the differential equation

$$w'' + 4w' + 20w = 8e^{-2t}\,.$$

(2) Give a real general solution to the differential equation

$$x'' - x' - 6x = 10e^{-2t}\,.$$

(3) Give a real general solution to the differential equation

$$y'' - y' - 6y = 13\cos(2t)\,.$$