Quiz 6 Solutions, Math 246, Professor David Levermore
Thursday, 8 March 2018

(1) [2] Given that e^{-t} is a particular solution of the equation

$$u'' + 6u' + 25u = 20e^{-t} \,,$$

give a real general solution.

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 + 6\mathrm{D} + 25$. Its characteristic polynomial is $p(z) = z^2 + 6z + 25 = (z+3)^2 + 4^2$, which has conjugate pair of roots $-3 \pm i4$. Therefore a real general solution of the associated homogeneous equation is

$$u_H(t) = c_1 e^{-3t}\cos(4t) + c_2 e^{-3t}\sin(4t) \,.$$

Because we are given that a particular solution is $u_P(t) = e^{-t}$, a real general solution of the nonhomogeneous equation is

$$u(t) = c_1 e^{-3t}\cos(4t) + c_2 e^{-3t}\sin(4t) + e^{-t} \,.$$

Remark. The forcing $20e^{-t}$ has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = -1$, and multiplicity $m = 0$. Therefore you should be able to show that e^{-t} is a particular solution of the equation by using either Key Identity Evaluations, the Zero Degree Formula, Undetermined Coefficients, or the Green Function. Try all four methods!

(2) [3] Give the degree, characteristic, and multiplicity for the forcing term of the equation

$$v'' + 6v' + 25v = 7t^5 e^{-3t}\cos(4t) \,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 + 6\mathrm{D} + 25$. Its characteristic polynomial is $p(z) = z^2 + 6z + 25 = (z+3)^2 + 4^2$, which has conjugate pair of roots $-3 \pm i4$.

The forcing term $7t^5 e^{-3t}\cos(4t)$ has degree $d = 5$, characteristic $\mu + i\nu = -3 + i4$, and multiplicity $m = 1$.

Group Work Questions for Problem 2 [4]

(a) Which derivatives of the Key Identity with respect to z are needed to find a particular solution of this equation by the method of Key Identity Evaluations? (Do not carry out the method!)
(b) If the method of Undetermined Coefficients is used to find a particular solution of this equation, what form will the solution take? (Do not carry out the method!)
(c) Compute the Green function $g(t)$ for the differential operator $\mathrm{L} = \mathrm{D}^2 + 6\mathrm{D} + 25$.
(d) Use the Green function to express the particular solution of $\mathrm{L}(v) = 7t^5 e^{-3t}\cos(4t)$ that satisfies the initial conditions $v(0) = v'(0) = 0$ in terms of two definite integrals. (Do not evaluate the integrals!)

(3) [5] Find a particular solution of the equation

$$w'' - 4w' + 4w = 10e^{2t}\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 - 4\mathrm{D} + 4$. Its characteristic polynomial is $p(z) = z^2 - 4z + 4 = (z-2)^2$, which has the double root 2.

Its forcing has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = 2$, and multiplicity $m = 2$. A particular solution $w_P(t)$ should be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Below we show that each of these methods gives the particular solution

$$w_P(t) = 5t^2 e^{2t}\,.$$

Remark. Because a real general solution of the associated homogeneous equation is

$$w_H(t) = c_1 e^{2t} + c_2 t\, e^{2t}\,,$$

a real general solution of the nonhomogeneous equation is

$$w(t) = c_1 e^{2t} + c_2 t\, e^{2t} + 5t^2 e^{2t}\,.$$

Key Identity Evaluations. Because $d = 0$ and $m = 2$, we need to evaluate the second derivative of the Key Identity at $z = \mu + i\nu = 2$. Because $p(z) = z^2 - 4z + 4$, the Key Identity and its first two derivatives with respect to z are

$$\begin{aligned}
\mathrm{L}(e^{zt}) &= (z^2 - 4z + 4)e^{zt}\,,\\
\mathrm{L}(t\, e^{zt}) &= (z^2 - 4z + 4)t\, e^{zt} + (2z - 4)e^{zt}\,,\\
\mathrm{L}(t^2 e^{zt}) &= (z^2 - 4z + 4)t^2 e^{zt} + 2(2z - 4)t\, e^{zt} + 2e^{zt}\,.
\end{aligned}$$

By evaluating the second derivative of the Key Identity at $z = 2$ we obtain

$$\mathrm{L}(t^2 e^{2t}) = (2^2 - 4\cdot 2 + 4)t^2 e^{zt} + 2(2\cdot 2 - 4)t\, e^{zt} + 2e^{zt} = 2e^{2t}\,.$$

Therefore a particular solution of $\mathrm{L}(w) = 10e^{2t}$ is

$$w_P(t) = 5t^2 e^{2t}\,.$$

Zero Degree Formula. This formula can be used because $d = 0$. For a forcing in the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)\,,$$

it gives the particular solution

$$w_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right)\,.$$

For this problem $f(t) = 10e^{2t}$ and $p(z) = z^2 - 4z + 4$, so that $\mu = 2$, $\nu = 0$, $\alpha = 10$, $\beta = 0$, $m = 2$, and $p''(z) = 2$, whereby

$$w_P(t) = t^2 e^{2t}\,\frac{10}{p''(2)} = \frac{10}{2}\,t^2 e^{2t} = 5t^2 e^{2t}\,.$$

Undetermined Coefficients. Because $m + d = 2$, $m = 2$, and $\mu + i\nu = 2$, there is a particular solution of $\mathrm{L}(w) = 10e^{2t}$ in the form

$$w_P(t) = At^2e^{2t}\,.$$

Because

$$w_P'(t) = A2t^2e^{2t} + A2t\,e^{2t}\,, \qquad w_P''(t) = A4t^2e^{2t} + 8At\,e^{2t} + A2\,e^{2t}\,,$$

we find that

$$\begin{aligned}\mathrm{L}\big(w_P(t)\big) &= w_P''(t) - 4w_P'(t) + 4w_P(t)\\ &= \big[A4t^2e^{2t} + 8At\,e^{2t} + A2\,e^{2t}\big] - 4\big[A2t^2e^{2t} + A2t\,e^{2t}\big] + 4At^2e^{2t}\\ &= 2Ae^{2t}\,.\end{aligned}$$

By setting $2Ae^{2t} = 10e^{2t}$ we see that $2A = 10$, whereby $A = 5$. Therefore a particular solution of $\mathrm{L}(w) = 8e^{2t}$ is

$$w_P(t) = 5t^2e^{2t}\,.$$

Group Work Questions for Problem 3 [2]

(a) Compute the Green function $g(t)$ for the differential operator $\mathrm{L} = \mathrm{D}^2 - 4\mathrm{D} + 4$.
(b) Find the particular solution of $\mathrm{L}(w) = 10e^{2t}$ that satisfies the initial conditions $w(0) = w'(0) = 0$.

Group Work Preview Questions for the Second Exam [4]

(1) A spring-mass system is governed by the initial-value problem

$$\ddot{h} + 2\eta\dot{h} + 9h = 0\,, \qquad h(0) = 1\,, \quad \dot{h}(0) = 0\,,$$

where $\eta \geq 0$.
(a) What is the natural frequency and period of the spring?
(b) For what value of η is the system undamped?
(c) For what values of η is the system under damped?
(d) For what value of η is the system critically damped?
(e) For what values of η is the system over damped?

(2) A spring-mass system is governed by the initial-value problem

$$\ddot{h} + 9h = 10\cos(\omega t)\,, \qquad h(0) = 0\,, \quad \dot{h}(0) = 0\,.$$

(a) Is this system undamped, under damped, critically damped, or over damped?
(b) For what value of ω does resonance occur?