Quiz 7 Solutions, Math 246, Professor David Levermore
Thursday, 29 March 2018

(1) [5] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for the function $f(t) = u(t-5)e^{-2t}$, where u is the unit step function.

Solution. By the definitions of the Laplace transform and the unit step function

$$\begin{aligned}\mathcal{L}[f](s) &= \lim_{T\to\infty}\int_0^T e^{-st} f(t)\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T e^{-st}u(t-5)e^{-2t}\,\mathrm{d}t \\ &= \lim_{T\to\infty}\int_5^T e^{-st}e^{-2t}\,\mathrm{d}t = \lim_{T\to\infty}\int_5^T e^{-(s+2)t}\,\mathrm{d}t\,.\end{aligned}$$

For $s \leq -2$ we have $e^{-(s+2)t} \geq 1$, so for $T > 5$

$$\lim_{T\to\infty}\int_5^T e^{-(s+2)t}\,\mathrm{d}t \geq \lim_{T\to\infty}\int_5^T \mathrm{d}t = T-5\,,$$

whereby $\mathcal{L}[f](s)$ is undefined for $s \leq -2$ because

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\int_5^T e^{-(s+2)t}\,\mathrm{d}t \geq \lim_{T\to\infty}(T-5) = \infty \qquad \text{for } s \leq -2\,.$$

For $s > -2$ and $T > 5$

$$\int_5^T e^{-(s+2)t}\,\mathrm{d}t = -\frac{e^{-(s+2)t}}{s+2}\bigg|_5^T = \frac{e^{-(s+2)5}}{s+2} - \frac{e^{-(s+2)T}}{s+2}\,,$$

whereby

$$\begin{aligned}\mathcal{L}[f](s) &= \lim_{T\to\infty}\int_5^T e^{-(s+2)t}\,\mathrm{d}t \\ &= \lim_{T\to\infty}\left[\frac{e^{-(s+2)5}}{s+2} - \frac{e^{-(s+2)T}}{s+2}\right] = \frac{e^{-(s+2)5}}{s+2} \qquad \text{for } s > -2\,.\end{aligned}$$

(2) [1] Give the exponential order as $t \to \infty$ of the function $u(t-5)t^2e^{-4t}\sin(3t)$.

Solution. The exponential orders as $t \to \infty$ of $u(t-5)$, t^2, e^{-4t}, and $\sin(3t)$ are 0, 0, -4, and 0 repsectively. Because the exponential order of a product is the sum of the exponential orders, we see that the exponential order as $t \to \infty$ of $u(t-5)t^2e^{-4t}\sin(3t)$ is $0 + 0 + (-4) + 0 = -4$.

Alternative Solution. The exponential order as $t \to \infty$ of $u(t-5)t^2e^{-4t}\sin(3t)$ is -4 because for every $\sigma > -4$ we have

$$\lim_{t\to\infty} e^{-\sigma t}u(t-5)t^2e^{-4t}\sin(3t) = \lim_{t\to\infty} t^2e^{-(\sigma+4)t}\sin(3t) = 0\,.$$

Group Work Exercises on Problems 1 and 2 [5]

- What is the exponential order of $f(t) = u(t-5)e^{-2t}$ as $t \to \infty$?
- How might we have guessed that $\mathcal{L}[f](s)$ is defined only for $s > -2$?
- If $h(t) = u(t-5)t^2e^{-4t}\sin(2t)$ then for what values of s will $\mathcal{L}[h](s)$ be defined?
- Use the table of Laplace transforms on the last page to compute $\mathcal{L}[h](s)$.

(3) [4] Find the Laplace transform $X(s)$ of the solution $x(t)$ of the initial-value problem

$$x'' + 16x = 0\,, \qquad x(0) = 2\,, \quad x'(0) = -4\,.$$

DO NOT solve for $x(t)$, just $X(s)$!

Solution. The Laplace transform of the initial-value problem gives

$$\mathcal{L}[x''](s) + 16\mathcal{L}[x](s) = 0\,,$$

where

$$\begin{aligned} \mathcal{L}[x](s) &= X(s)\,, \\ \mathcal{L}[x'](s) &= s\mathcal{L}[x](s) - x(0) = sX(s) - 2\,, \\ \mathcal{L}[x''](s) &= s\mathcal{L}[x'](s) - x'(0) = s\big(sX(s) - 2\big) - (-4) = s^2X(s) - 2s + 4\,. \end{aligned}$$

By placing these into the Laplace transform of the initial-value problem gives

$$\big(s^2X(s) - 2s + 4\big) + 16X(s) = 0\,,$$

which yields

$$(s^2 + 16)X(s) - 2s + 4 = 0\,,$$

whereby

$$X(s) = \frac{2s - 4}{s^2 + 16}\,.$$

Group Work Exercises on Problem 3 [5]

- Use the table below to compute $x(t) = \mathcal{L}^{-1}[X](t)$.
- Use the Laplace transform to compute the Green function $g(t)$ for the operator $\mathrm{D}^2 + 16$.
- Compute the natural fundamental set of solutions $N_0(t)$, $N_1(t)$ associated with initial time 0 for the operator $\mathrm{D}^2 + 16$.
- Use the Laplace transform to solve the initial-value problem
$$h'' + 16h = 24u(t-3)\sin(2(t-3))\,, \qquad h(0) = 0\,, \quad h'(0) = 0\,.$$

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2 + b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2 + b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)$$

and u is the unit step function.