Quiz 8 Solutions, Math 246, Professor David Levermore
Thursday, 5 April 2018

Short Table: $\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$ for $s > a$, $\qquad \mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$.

(1) [3] The Green function for the linear differential operator $\mathrm{D}^2 + 2\mathrm{D} + 1$ is $g(t) = t\,e^{-t}$. Find its natural fundamental set of solutions associated with $t_I = 0$.

Solution. Because the characteristic polynomial of $\mathrm{D}^2 + 2\mathrm{D} + 1$ is $p(s) = s^2 + 2s + 1$, the natural fundamental set of solutions associated with $t_I = 0$ is

$$N_1(t) = g(t) = t\,e^{-t}\,,$$

$$N_0(t) = \mathrm{D}N_1(t) + 2g(t) = \mathrm{D}(t\,e^{-t}) + 2t\,e^{-t} = e^{-t} - t\,e^{-t} + 2t\,e^{-t} = e^{-t} + t\,e^{-t}\,.$$

(2) [3] Find $V(s) = \mathcal{L}[v](s)$ where $v(t) = u(t-3)e^{-4t}$.

Solution. To use the second entry of our table we write

$$v(t) = u(t-3)e^{-4t} = u(t-3)j(t-3)\,,$$

which implies that $j(t-3) = e^{-4t}$, whereby the shifty step method gives

$$j(t) = e^{-4(t+3)} = e^{-12}e^{-4t}\,.$$

By the second entry in our table with $c = 3$ and the first entry with $n = 0$ and $a = -4$ we find that

$$\begin{aligned} V(s) &= \mathcal{L}[v](s) = \mathcal{L}[u(t-3)j(t-3)](s) = e^{-3s}\mathcal{L}[j](s) \\ &= e^{-3s}\mathcal{L}[e^{-12}e^{-4t}](s) = e^{-12}e^{-3s}\mathcal{L}[e^{-4t}](s) = e^{-12}e^{-3s}\,\frac{1}{s+4}\,. \end{aligned}$$

(3) [4] Find $y(t) = \mathcal{L}^{-1}[Y](t)$ where $Y(s) = e^{-2s}\dfrac{10}{(s-4)(s+1)}$.

Solution. By the partial fraction identity

$$J(s) = \frac{10}{(s-4)(s+1)} = \frac{2}{s-4} + \frac{-2}{s+1}\,,$$

and by the first entry in our table with $n = 0$ and $a = 4$ and with $n = 0$ and $a = -1$ we have

$$\begin{aligned} j(t) &= \mathcal{L}^{-1}\left[\frac{10}{(s-4)(s+1)}\right](t) \\ &= 2\mathcal{L}^{-1}\left[\frac{1}{s-4}\right](t) - 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = 2e^{4t} - 2e^{-t}\,. \end{aligned}$$

Therefore by the second entry in our table we have

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y](t) = \mathcal{L}^{-1}\left[e^{-2s}J(s)\right](t) = u(t-2)j(t-2) \\ &= u(t-2)\big(2e^{4(t-2)} - 2e^{-(t-2)}\big) = 2u(t-2)\big(e^{4t-8} - e^{-t+2}\big)\,. \end{aligned}$$

A Longer Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s) \text{ and } u \text{ is the unit step function}\,.$$

Group Work Exercises for Problems 2 and 3 [5]

(1) Find $x(t) = \mathcal{L}^{-1}[X](t)$ where $X(s) = e^{-2s}\dfrac{10}{(s^2-4)(s^2+1)}$.

(2) Find $y(t) = \mathcal{L}^{-1}[Y](t)$ where $Y(s) = e^{-3s}\dfrac{2s+14}{s^2+4s+29}$.

(3) Find $F(s) = \mathcal{L}[f](s)$ where $f(t) = u(t-3)e^{-4t}\cos(2t) + 7\delta(t-5)$.

(4) Find $F(s) = \mathcal{L}[f](s)$ where $f(t) = u(t-3)t^2e^{-4t}\cos(2t)$.

(5) Find $F(s) = \mathcal{L}[f](s)$ where

$$f(t) = \begin{cases} \cos(2t) & \text{for } 0 \le t < \pi\,, \\ 1 & \text{for } \pi \le t < 5\,, \\ e^{5-t} & \text{for } 5 \le t < \infty\,. \end{cases}$$

Group Work Exercises for Quiz 9 Preview [5]

(1) [1] Transform the equation $v'''' + v^3 v''' - (v')^4 - \sin(t+v) = e^t$ into a first-order system of ordinary differential equations.

(2) [2] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 23 liters and the second contains 32 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 17 grams of salt in the first tank and 29 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(3) [2] Consider the matrix-valued function $\boldsymbol{\Psi}(t) = \begin{pmatrix} 1 & -2t \\ t^3 & 4-t^4 \end{pmatrix}$.

(a) Compute $\boldsymbol{\Psi}(t)^{-1}$.
(b) Compute $\boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(t)^{-1}$.