Quiz 9 Solutions, Math 246, Professor David Levermore
Thursday, 12 April 2018

(1) [2] Transform the equation $y''' - y^2 y'' + (y')^3 - \cos(t+y) = e^y$ into a first-order system of ordinary differential equations.

Solution. Because the equation is third order, the first-order system must have dimension three. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ e^{x_1} + x_1^2 x_3 - x_2^3 + \cos(t + x_1) \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}.$$

(2) [3] Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{4e^t}{25 - t^2} & \dfrac{3}{t} \\ \dfrac{3}{t} & \dfrac{4e^t}{25 - t^2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \qquad \begin{pmatrix} u(-3) \\ v(-3) \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \end{pmatrix}.$$

Solution. This first-order homogeneous linear system is already in *normal form.* Notice the following:

- ⋄ the coefficients $4e^t/(25 - t^2)$ are undefined at $t = \pm 5$ and continuous elsewhere;
- ⋄ the coefficients $3/t$ are undefined at $t = 0$ and continuous elsewhere;
- ⋄ the initial time is $t_I = -3$.

Therefore the interval of definition for the solution of the initial-value problem is $(-5, 0)$ because:

- the initial time $t_I = -3$ lies within $(-5, 0)$,
- all the coefficients are continuous over $(-5, 0)$,
- the coefficients $4e^t/(25 - t^2)$ are undefined at $t = -5$,
- the coefficients $3/t$ are undefined at $t = 0$.

Remark. On exams the reasoning given by the bullets must be given for full credit! The first two bullets give the reasons why the interval of definition is at least $(-5, 0)$. The second two bullets give the reasons why the interval of definition is not larger.

Group Work Exercises for Problem 2 [2]

(a) Give the interval of definition if the initial time is $t_I = 2$.

(b) Give the interval of definition if the initial time is $t_I = 6$.

(c) Give the interval of definition if the initial time is $t_I = -8$.

(3) [5] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} e^t \\ -t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} e^t \\ 1 \end{pmatrix}$.

(a) [2] Compute their Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.

Solution (a).

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} e^t & e^t \\ -t^2 & 1 \end{pmatrix} = e^t \cdot 1 - (-t^2) \cdot e^t = e^t + t^2 e^t = (1+t^2)e^t \,.$$

Solution (b). Let $\mathbf{\Psi}(t) = \begin{pmatrix} e^t & e^t \\ -t^2 & 1 \end{pmatrix}$. Because $\mathbf{\Psi}'(t) = \mathbf{A}(t)\mathbf{\Psi}(t)$, we have

$$\begin{aligned} \mathbf{A}(t) = \mathbf{\Psi}'(t)\, \mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} e^t & e^t \\ -2t & 0 \end{pmatrix} \begin{pmatrix} e^t & e^t \\ -t^2 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{(1+t^2)e^t} \begin{pmatrix} e^t & e^t \\ -2t & 0 \end{pmatrix} \begin{pmatrix} 1 & -e^t \\ t^2 & e^t \end{pmatrix} = \frac{1}{(1+t^2)e^t} \begin{pmatrix} (1+t^2)e^t & 0 \\ -2t & 2te^t \end{pmatrix} . \end{aligned}$$

Remark. The matrix multiplication had to be carried out for full credit.

Group Work Exercises for Problem 3 [4]

Consider the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ that was found in part (b).

(a) Give a general solution of this system.

(b) Give a fundamental matrix for this system.

(c) Find the natural fundamental matrix of this system for $t_I = 0$.

(d) Find the natural fundamental matrix of this system for $t_I = 1$.

(e) Find the Green matrix $\mathbf{G}(t,s)$ associated with this system.

Group Work Exercises for Quiz 10 Preview [4]

(1) Suppose $e^{t\mathbf{A}} = \begin{pmatrix} \cos(t) & -\frac{1}{2}\sin(t) \\ 2\sin(t) & \cos(t) \end{pmatrix}$.

(a) Give a general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(b) Find the natural fundamental matrix of this system for $t_I = 0$.

(c) Find the natural fundamental matrix of this system for $t_I = \frac{\pi}{2}$.

(d) Find the Green matrix $\mathbf{G}(t,s)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$.

(e) Find $\mathbf{A}$.

(2) Let $\mathbf{A} = \begin{pmatrix} -3 & 4 \\ -1 & -3 \end{pmatrix}$.

(a) Find $e^{t\mathbf{A}}$.

(b) Give a general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(c) Find the natural fundamental matrix of this system for $t_I = 0$.

(d) Find the natural fundamental matrix of this system for $t_I = 2$.

(e) Find the Green matrix $\mathbf{G}(t,s)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$.