**Quiz 10 Solutions, Math 246, Professor David Levermore**
**Thursday, 19 April 2018**

(1) [4] Let $\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 4 & -5 \end{pmatrix}$. Compute $e^{t\mathbf{A}}$.

**Solution.** Because $\mathbf{A}$ is $2 \times 2$ its characteristic polynomial is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 6z + 13 = (z+3)^2 + 4\,,$$

which has the conjugate pair of roots $-3 \pm i2$. Therefore

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-3t}\left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A} + 3\mathbf{I})\right] \\ &= e^{-3t}\left[\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2}\begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}\right] \\ &= e^{-3t}\begin{pmatrix} \cos(2t) + \sin(2t) & -\sin(2t) \\ 2\sin(2t) & \cos(2t) - \sin(2t) \end{pmatrix}. \end{aligned}$$

**Group Work Exercises for Problem 1** [2]

(a) Give a general solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(b) Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(2) [2] The matrix $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ has eigenvalues 1 and 3.
Find an eigenvector associated with the eigenvalue 3.

**Solution.** An eigenvector $\mathbf{v}$ associated with the eigenvalue 3 is $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, because it is proportional to a nonzero column of

$$\mathbf{A} - 1\mathbf{I} = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}.$$

**Remarks.** Any nonzero multiple of $\mathbf{v}$ is an equally correct answer. The approach taken above is based upon the Cayley-Hamilton Theorem, and works only for $2 \times 2$ matrices. An approach taught in linear algebra courses that works for any size matrix is to solve $(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \mathbf{0}$, but this approach takes longer than the one taken above.

**Group Work Exercises for Problem 2** [3]

(a) Find an eigenvector of $\mathbf{B}$ associated with the eigenvalue 1.

(b) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{B} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$.

(c) Use $\mathbf{V}$ and $\mathbf{D}$ to compute $e^{t\mathbf{B}}$.

(3) [4] The $2 \times 2$ matrix $\mathbf{C}$ has the eigenpairs

$$\left(2\,,\begin{pmatrix}1\\-1\end{pmatrix}\right)\,,\qquad \left(-1\,,\begin{pmatrix}1\\1\end{pmatrix}\right)\,.$$

Give a fundamental matrix for the system $\mathbf{y}' = \mathbf{C}\mathbf{y}$.

**Solution.** The eigensolutions constructed from the eigenpairs are

$$\mathbf{y}_1(t) = e^{2t}\begin{pmatrix}1\\-1\end{pmatrix}\,,\qquad \mathbf{y}_2(t) = e^{-t}\begin{pmatrix}1\\1\end{pmatrix}\,.$$

Therefore a fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{pmatrix}\mathbf{y}_1(t) & \mathbf{y}_2(t)\end{pmatrix} = \begin{pmatrix}e^{2t} & e^{-t}\\ -e^{2t} & e^{-t}\end{pmatrix}\,.$$

**Group Work Exercises for Problem 3** [5]

(a) Use $\mathbf{\Psi}(t)$ to compute $e^{t\mathbf{C}}$.

(b) Find $\mathbf{C}$.

(c) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$.

(d) Use $\mathbf{V}$ and $\mathbf{D}$ to compute $e^{t\mathbf{C}}$.

(e) Compute $(s\mathbf{I} - \mathbf{C})^{-1}$ wherever it is defined.