Quiz 11 Solutions, Math 246, Professor David Levermore
Tuesday, 1 May 2018

(1) [5] The eigenpairs of a 2×2 matrix $\mathbf{A}$ are

$$\left(1\,,\begin{pmatrix}1\\2\end{pmatrix}\right)\,,\qquad \left(3\,,\begin{pmatrix}2\\-1\end{pmatrix}\right)\,.$$

(a) [2] Classify the phase-plane portrait of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
(b) [2] Sketch the phase-plane portrait of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
(c) [1] Determine the stability of the origin for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution (a). Because $\mathbf{A}$ has two positive eigenvalues, the phase-plane portrait of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a *nodal source*.

Solution (b). Your sketch should show the line through the origin and the point $(1, 2)$, and the line through the origin and the point $(2, -1)$. Each line should have arrows on it pointing away from the origin. These two lines split the plane into four regions. Your sketch should show at least one representative orbit in each of these four regions. Each representative orbit should emerge from the origin tangent to the line through the point $(1, 2)$ and curve to become more parallel to the line through the point $(2, -1)$ as it gets further away from the origin. Each representative orbit should have arrows on it pointing away from the origin.

Solution (c). Because the phase-plane portrait of the system is a nodal source, the origin is *repelling*.

(2) [5] Consider the planar system

$$\mathbf{x}' = \mathbf{B}\mathbf{x}\,, \qquad \text{where} \quad \mathbf{B} = \begin{pmatrix}-3 & 4\\ -2 & 1\end{pmatrix}\,.$$

(a) [2] Classify its phase-plane portrait.
(b) [2] Sketch its phase-plane portrait.
(c) [1] Determine the stability of the origin for this system.

Solution (a). The characteristic polynomial of $\mathbf{B}$ is

$$\begin{aligned}p(z) = z^2 - \operatorname{tr}(\mathbf{B})z + \det(\mathbf{B}) &= z^2 - (-2)z + \big((-3)1 - (-2)4\big)\\ &= z^2 + 2z + 5 = (z+1)^2 + 2^2\,.\end{aligned}$$

Because this has the conjugate pair of roots $-1 \pm i2$, the phase-plane portrait of the system $\mathbf{x}' = \mathbf{B}\mathbf{x}$ is a spiral sink. Because the b_{21} entry is negative, the phase-plane portrait is a *clockwise spiral sink*.

Solution (b). Your sketch should show a curve that spirals into the origin in a clockwise fashion.

Solution (c). Because the phase-plane portrait of the system is a spiral sink, the origin is *attracting*.

Group Work Exercises based on Quiz 11 [4]

(1) [2] The eigenpairs of a 2×2 matrix $\mathbf{C}$ are

$$\left(1\,,\, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)\,, \qquad \left(-3\,,\, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)\,.$$

(a) Classify and sketch the phase-plane portrait of the system $\mathbf{x}' = \mathbf{C}\mathbf{x}$.
(b) Determine the stability of the origin for the system $\mathbf{x}' = \mathbf{C}\mathbf{x}$.

(2) [2] Consider the planar system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}\,, \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\,.$$

(a) Classify and sketch the phase-plane portrait of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
(b) Determine the stability of the origin for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Group Work Exercises based on the Lectures [6]

(1) [2] Consider the planar system

$$u' = (5 - v)u\,, \qquad v' = (u - 2)v\,.$$

(a) Find all of its stationary solutions.
(b) Find all of its semistationary solutions.
(c) Find a nonconstant function $H(u, v)$ such that every orbit of this system satisfies $H(u, v) = c$ for some constant c.
(d) Sketch the phase-plane portrait of this system.

(2) [2] Consider the planar system

$$x' = 2x - y\,, \qquad y' = -2y + x^2\,.$$

(a) Find all of its stationary solutions.
(b) Find all of its semistationary solutions.
(c) Find a nonconstant function $H(x, y)$ such that every orbit of this system satisfies $H(x, y) = c$ for some constant c.
(d) Sketch the phase-plane portrait of this system.

(3) [2] Consider the planar system

$$p' = -2p + q\,, \qquad q' = -3q + 2p^2\,.$$

(a) Find all of its stationary solutions.
(b) Find all of its semistationary solutions.
(c) Classify the type and stability of each stationary point.
(d) Sketch the phase-plane portrait of this system.