Quiz 12 Solutions, Math 246, Professor David Levermore
Tuesday, 8 May 2018

(1) [5] Consider the system

$$x' = 2x + y\,, \qquad y' = -x - 2y - x^2\,.$$

(a) [2] Find all of its stationary points.
(b) [3] Find a nonconstant function $H(x,y)$ such that every orbit of this system satisfies $H(x,y) = c$ for some constant c.

Solution (a). The stationary points satisfy

$$0 = 2x + y\,, \qquad 0 = -x - 2y - x^2\,.$$

The first equation is satisfied if and only if $y = -2x$, whereby the second equation becomes $0 = 3x - x^2$, which is solved by $x = 0$ or $x = 3$. Therefore all the stationary points are $(0,0)$ and $(3,-6)$.

Solution (b). The system is Hamiltonian because

$$\partial_x(2x + y) + \partial_y\big(-x - 2y - x^2\big) = 2 - 2 = 0\,.$$

Therefore there exists $H(x,y)$ such that

$$\partial_y H(x,y) = 2x + y\,, \qquad -\partial_x H(x,y) = -x - 2y - x^2\,.$$

By integrating the first equation we find $H(x,y) = 2xy + \frac{1}{2}y^2 + h(x)$. By substituting this into the second equation we see that $-2y - h'(x) = -x - 2y - x^2$, whereby $h'(x) = x + x^2$. Therefore we can set

$$H(x,y) = 2xy + \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\,.$$

(2) [5] Consider the system

$$u' = 3u - v\,, \qquad v' = 2v - 3u^2\,.$$

Its stationary points are $(0,0)$ and $(2,6)$. Classify the type and stability of each of these stationary points. (You do not have to sketch anything.)

Solution. The Jacobian matrix is $\boldsymbol{\partial}\mathbf{f}(u,v) = \begin{pmatrix} \partial_u f & \partial_v f \\ \partial_u g & \partial_v g \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -6u & 2 \end{pmatrix}$.

- At $(0,0)$ the coefficient matrix of its linearization is $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(0,0) = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$. Because $\mathbf{A}$ is upper triangular, we can see that its eigenvalues are 3 and 2. Because these eigenvalues are both positive, the stationary point $(0,0)$ is a *nodal source* and thereby is *repelling*.
- At $(2,6)$ the coefficient matrix of its linearization is $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(2,6) = \begin{pmatrix} 3 & -1 \\ -12 & 2 \end{pmatrix}$, which has characteristic polynomial $p(z) = z^2 - 5z - 6 = (z-6)(z+1)$. The eigenvalues of $\mathbf{A}$ are 6 and -1. Because these are real, nonzero, and have opposite sign, the stationary point $(2,6)$ is a *saddle* and thereby is *unstable*, but not repelling.