Introduction to Singular Integral Operators

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History

Starting in the 1800s integral operators arose in many settings.

- **Niels Henrik Abel** (1823)
- **Carl G. Neumann** (1877)
- **Giuseppe Peano** (1886, 1890)
- **Émile Picard** (1890)
- **Ernst Lindelöf** (1894)
- **William F. Osgood** (1898)

History

General theories were developed in the early 1900s.

- **Ivar Fredholm** (1900, 1903)
- **David Hilbert** (1904)
- **Erhard Schmidt** (1907)

These focused on integral equations in the form

$$u(x) - \int_a^b k(x, y) u(y) \, \mathrm{d}y = f(x) \,,$$

where $[a, b] \subset \mathbb{R}$, and the kernel $k(x, y)$ is continuous over $[a, b]^2$, and forcing $f(x)$ is continuous over $[a, b]$. Hilbert and Schmidt extended their results to some so-called *singular* kernels.

History

Major advances in the theory of singular integral operators followed.

- **William H. Young** (1912)
- **Godfrey H. Hardy, John E. Littlewood** (1928, 1930)
- **Sergei Sobolev** (1938)
- **Alberto Cauldrón, Antoni Zygmund** (1956)

Here we will present some extensions of these advances.

Integral Operators

Let $(X, \Sigma_\mu, \mathrm{d}\mu)$ and $(Y, \Sigma_\nu, \mathrm{d}\nu)$ be positive σ-finite measure spaces.

Let $M(\mathrm{d}\mu)$ and $M(\mathrm{d}\nu)$ be the spaces of all complex-valued $\mathrm{d}\mu$-measurable and $\mathrm{d}\nu$-measurable functions respectively.

We consider linear integral operators $\mathcal{K}$ in the form

$$\mathcal{K}u(y) = \int k(x, y)\, u(x)\, \mathrm{d}\mu(x)\,, \tag{1.1}$$

where the kernel k is a complex-valued measurable function with respect to the σ-algebra $\Sigma_{\mu\otimes\nu}$.

Integral Operators

We give conditions on k that imply the operator $\mathcal{K}$ is bounded or even compact from $\mathcal{X}$ to $\mathcal{Y}$ where $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ are Banach spaces of functions that are contained within $M(\mathrm{d}\mu)$ and $M(\mathrm{d}\nu)$ respectively.

We will first do this for classical Lebesgue spaces — namely, for cases where

$$\mathcal{X} = L^p(\mathrm{d}\mu) \text{ and } \mathcal{Y} = L^{q^*}(\mathrm{d}\nu) \text{ for some } p, q^* \in [1, \infty] \,.$$

We will then extend these results to weak Lebesgue spaces — namely, to cases where

$$\mathcal{X} = L^p_w(\mathrm{d}\mu) \text{ or } \mathcal{Y} = L^{q^*}_w(\mathrm{d}\nu) \text{ for some } p, q^* \in (1, \infty) \,.$$

Lebesgue Spaces

For any positive σ-finite measure space $(X, \Sigma_\mu, \mathrm{d}\mu)$ and any $p \in (0, \infty)$ we define the *Lebesgue space* $L^p(\mathrm{d}\mu)$ by

$$L^p(\mathrm{d}\mu) = \left\{ u \in M(\mathrm{d}\mu) \,:\, \int |u(x)|^p \,\mathrm{d}\mu(x) < \infty \right\}. \tag{2.2}$$

For every $p \in (0, \infty)$ we define the magnitude of $u \in L^p(\mathrm{d}\mu)$ by

$$[u]_{L^p} = \left(\int |u(x)|^p \,\mathrm{d}\mu(x) \right)^{\frac{1}{p}}. \tag{2.3}$$

It is clear from (2.2) that for every $u \in M(\mathrm{d}\mu)$ we have $u \in L^p(\mathrm{d}\mu)$ if and only if $[u]_{L^p} < \infty$.

It is also clear that $[\lambda\, u]_{L^p} = |\lambda|\, [u]_{L^p}$ for every $u \in L^p(\mathrm{d}\mu)$ and every $\lambda \in \mathbb{C}$.

Lebesgue Spaces

For every $p \in [1, \infty)$ the Minkowski inequality implies that $[\,\cdot\,]_{L^p}$ satisfies the triangle inequality, and is thereby a norm. In that case $L^p(\mathrm{d}\mu)$ is a Banach space equipped with the norm

$$\|u\|_{L^p} = [u]_{L^p} = \left(\int |u(x)|^p \, \mathrm{d}\mu(x) \right)^{\frac{1}{p}} . \tag{2.4}$$

For every $p \in (0, 1)$ it can be shown that $[\,\cdot\,]_{L^p}$ fails to satisfy the triangle inequality, and is thereby not a norm. However, in that case $L^p(\mathrm{d}\mu)$ is a Frechét space equipped with the metric

$$\mathrm{d}(u, v)_{L^p} = [u - v]_{L^p}^p = \int |u(x) - v(x)|^p \, \mathrm{d}\mu(x) . \tag{2.5}$$

Lebesgue Spaces

Finally, we define the *Lebesgue space* $L^\infty(\mathrm{d}\mu)$ by

$$L^\infty(\mathrm{d}\mu) = \left\{ u \in M(\mathrm{d}\mu) \, : \, \operatorname*{ess\,sup}_{x \in X} \{|u(x)|\} < \infty \right\} . \tag{2.6}$$

Then $L^\infty(\mathrm{d}\mu)$ is a Banach space equipped with the norm

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in X} \{|u(x)|\} = \inf \{\alpha > 0 \, : \, \mu(E_u(\alpha)) = 0\} , \tag{2.7}$$

where $E_u(\alpha) = \{x \in X \, : \, |u(x)| > \alpha\}$. Here we adopt the usual convention that $\inf\{\emptyset\} = \infty$.

Weak Lebesgue Spaces

Early work did not apply to kernels over $\mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}$ of the form $k(x, y) = |x - y|^{-\frac{\mathrm{D}}{r}}$ for some $r \in (1, \infty)$ when $\mathrm{d}\mu$ and $\mathrm{d}\nu$ are each Lebesgue measure. This problem was overcome by bounds that grew out of the pioneering work of Hardy and Littlewood. Their work led to a class of spaces that allow the treatment of such kernels — namely, the weak Lebesgue spaces.

For any positive σ-finite measure space $(X, \Sigma_\mu, \mathrm{d}\mu)$ and any $p \in (0, \infty)$ we define the *weak Lebesgue space* $L^p_w(\mathrm{d}\mu)$ by

$$L^p_w(\mathrm{d}\mu) = \left\{ u \in M(\mathrm{d}\mu) \;:\; \sup_{\alpha>0} \left\{ \alpha^p \mu(E_u(\alpha)) \right\} < \infty \right\}, \qquad (2.8)$$

where $E_u(\alpha) = \{x \in X \;:\; |u(x)| > \alpha\}$.

Weak Lebesgue Spaces

For every $p \in (0, \infty)$ it is clear that $L^p(\mathrm{d}\mu) \subset L^p_w(\mathrm{d}\mu)$. Indeed, for every $u \in L^p(\mathrm{d}\mu)$ and every $\alpha > 0$ the Chebyshev inequality yields

$$\mu(E_u(\alpha)) = \int_{E_u(\alpha)} \mathrm{d}\mu(x) \leq \frac{1}{\alpha^p} \int_{E_u(\alpha)} |u(x)|^p \, \mathrm{d}\mu(x) \leq \frac{1}{\alpha^p} \int |u(x)|^p \, \mathrm{d}\mu(x) =$$

It thereby follows that

$$\sup_{\alpha>0} \left\{ \alpha^p \mu(E_u(\alpha)) \right\} \leq [u]^p_{L^p(\mathrm{d}\mu)} < \infty \,,$$

whereby $u \in L^p_w(\mathrm{d}\mu)$. In general $L^p_w(\mathrm{d}\mu)$ is larger than $L^p(\mathrm{d}\mu)$. For example, when $X = \mathbb{R}^{\mathrm{D}}$ and $\mathrm{d}\mu$ is the unsual Lebesgue measure on $\mathbb{R}^{\mathrm{D}}$ then it can be shown that the function $u(x) = |x|^{-\frac{D}{p}}$ is in $L^p_w(\mathrm{d}\mu)$ but it is clearly not in $L^p(\mathrm{d}\mu)$.

Weak Lebesgue Spaces

For every $p \in (0, \infty)$ we define the magnitude of every $u \in L^p_w(\mathrm{d}\mu)$ by

$$[u]_{L^p_w} = \left(\sup_{\alpha > 0} \{ \alpha^p \mu(E_u(\alpha)) \} \right)^{\frac{1}{p}} . \tag{2.9}$$

It is clear from (2.8) that $u \in L^p_w(\mathrm{d}\mu)$ if and only if $[u]_{L^p_w} < \infty$.

However, $[\,\cdot\,]_{L^p_w}$ is not a norm. While it satisfies $[\lambda u]_{L^p_w} = |\lambda| \, [u]_{L^p_w}$ for every $u \in L^p_w(\mathrm{d}\mu)$ and $\lambda \in \mathbb{C}$, it fails to satisfy the triangle inequality.

However, the next result shows there is an equivalent norm for $p \in (1, \infty)$.

Weak Lebesgue Spaces

Theorem

For every $p \in (1,\infty)$ *and every* $u \in M(\mathrm{d}\mu)$ *we define*

$$\|u\|_{L^p_w} = \sup_{E\in\Sigma_\mu} \left\{ \frac{1}{\mu(E)^{\frac{1}{p^*}}} \int_E |u(x)|\,\mathrm{d}\mu(x) \,:\, \mu(E)\in(0,\infty) \right\}. \tag{2.10}$$

For every $u \in M(\mathrm{d}\mu)$ *we can show that* $u \in L^p_w(\mathrm{d}\mu)$ *if and only if* $\|u\|_{L^p_w} < \infty$. *Moreover,*

$$[u]_{L^p_w} \leq \|u\|_{L^p_w} \leq p^*[u]_{L^p_w} \quad \text{for every } u \in L^p_w(\mathrm{d}\mu). \tag{2.11}$$

Weak Lebesgue Spaces

Remark. It is easily checked from definition (2.10) that $\| \cdot \|_{L^p_w}$ is a norm. The result stated above shows that the space $L^p_w(\mathrm{d}\mu)$ is characterized by the finiteness of this norm for every $p \in (1, \infty)$.

Remark. If $u \in L^p(\mathrm{d}\mu)$ for some $p \in (1, \infty)$ then by applying the Hölder inequality inside the sup of (2.10) and using the fact that $\|\mathbf{1}_E\|_{L^{p^*}} = \mu(E)^{\frac{1}{p^*}}$ shows that

$$\|u\|_{L^p_w} \leq \|u\|_{L^p} .$$

Here $\mathbf{1}_E$ denotes the indicator function of the set E.

Convolution Operators

Let $(G, +)$ be an Abelian group with Haar measure $\mathrm{d}m$ defined over the σ-algebra Σ_m. (Recall that the Haar measure is a positive measure that is translation invariant; it is unique up to a positive constant factor.) Given two functions w and u defined over G, we define their *convolution* to be the function $w * u$ that is formally given by

$$w * u(y) = \int w(y - x)\, u(x)\, \mathrm{d}m(x)\,. \tag{3.12}$$

This can be viewed as an integral operator of the form (1.1) where $X = Y = G$, $\mathrm{d}\mu = \mathrm{d}\nu = \mathrm{d}m$, $\Sigma_\mu = \Sigma_\nu = \Sigma_m$ and $k(x, y) = w(y - x)$. Such operators are called *convolution operators*. In this setting, w is called the *convolution kernel*.

Here we give bounds that insure the convolution (3.12) maps between either classical Lebesgue spaces or weak Lebesgue spaces.

Young Convolution Inequality

We begin with the classical Young convolution inequality.

Theorem

Let $p, q, r \in [1, \infty]$ satisfy the relation

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2\,. \tag{3.13}$$

For every $u \in L^p(\mathrm{d}m)$, $v \in L^q(\mathrm{d}m)$, and $w \in L^r(\mathrm{d}m)$ we have

$$\iint \left| w(y-x)\, u(x)\, \overline{v(y)} \right| \mathrm{d}m(x)\, \mathrm{d}m(y) \leq \|u\|_{L^p}\, \|v\|_{L^q}\, \|w\|_{L^r}\,. \tag{3.14}$$

Hardy-Littlewood-Sobolev Inequalities

In the Young convolution inequality (3.14) the function w sits in $L^r(\mathrm{d}m)$. When $r \in (1, \infty)$ the Hardy-Littlewood-Sobolev inequalities allows this class to be extended to $L^r_w(\mathrm{d}m)$. The first Hardy-Littlewood-Sobolev inequality is as follows.

Theorem

Let $r \in (1, \infty)$. For every $u \in L^1(\mathrm{d}m)$ and $w \in L^r_w(\mathrm{d}m)$ we have

$$\|w * u\|_{L^r_w} \leq \|u\|_{L^1} \|w\|_{L^r_w} . \tag{3.15}$$

Hardy-Littlewood-Sobolev Inequalities

The second Hardy-Littlewood-Sobolev inequality is as follows.

Theorem

Let $p, q, r \in (1, \infty)$ that satisfy relation (3.13). Then there exists a positive constant $C^{p,q,r}_{G,w}$ such that for every $u \in L^p_w(\mathrm{d}m)$ and $w \in L^r_w(\mathrm{d}m)$ we have

$$\|w * u\|_{L^{q^*}_w} \leq C^{p,q,r}_{G,w} \, [u]_{L^p_w} \, [w]_{L^r_w} \,. \tag{3.16}$$

We can establish (3.16) with

$$C^{p,q,r}_{G,w} = \frac{p^* \, q^* \, r^*}{p \;\, q} \,. \tag{3.17}$$

Hardy-Littlewood-Sobolev Inequalities

The third Hardy-Littlewood-Sobolev inequality is as folows.

Theorem

Let $p, q, r \in (1, \infty)$ that satisfy relation (3.13). Then there exists a positive constant $C_G^{p,q,r}$ such that for every $u \in L^p(\mathrm{d}m)$, $v \in L^q(\mathrm{d}m)$, and $w \in L^r_w(\mathrm{d}m)$ we have

$$\iint \left| w(y-x)\, u(x)\, \overline{v(y)} \right| \mathrm{d}m(x)\, \mathrm{d}m(y) \leq C_G^{p,q,r} \, \|u\|_{L^p} \, \|v\|_{L^q} \, [w]_{L^r_w} \, . \quad (3.18)$$

We can establish (3.18) with

$$C_G^{p,q,r} = \frac{r^*}{pq} \left(\frac{p^*}{r} \right)^{\frac{1}{r} + \frac{r}{p^* r^*}} \left(\frac{q^*}{r} \right)^{\frac{1}{r} + \frac{r}{q^* r^*}} \leq \frac{p^* \, q^* \, r^*}{p \;\, q \;\, r^2} \, . \quad (3.19)$$

Calderon-Zygmund Inequality

We specialize to the case in which $G = \mathbb{R}^{\mathrm{D}}$ and $\mathrm{d}m$ is the usual Lebesgue measure on $\mathbb{R}^{\mathrm{D}}$. Calderon-Zygmund theory implies the following.

Let w be a complex-valued function over $\mathbb{R}^{\mathrm{D}}$ that has the factored form

$$w(z) = h(|z|)\, j\!\left(\frac{z}{|z|}\right) , \tag{3.20}$$

where h is Lipschitz continuous away from $z = 0$ and satisfies $\sup\{|z|^{\mathrm{D}}|h(|z|)| \,:\, |z| > 0\} < \infty$, while j is Lipschitz continuous over $\mathbb{S}^{\mathrm{D}-1}$ and satisfies the cancellation condition

$$\int_{\mathbb{S}^{\mathrm{D}-1}} j(o)\, \mathrm{d}S(o) = 0 . \tag{3.21}$$

Here $\mathrm{d}S$ denotes the usual Lebesgue surface measure on $\mathbb{S}^{\mathrm{D}-1}$. For every $\epsilon > 0$ define w_ϵ by $w_\epsilon(z) = \mathbf{1}_{\{|z|>\epsilon\}}\, w(z)$, and $\mathcal{K}_\epsilon$ by

$$\mathcal{K}_\epsilon u(y) = \int w_\epsilon(y - x)\, u(x)\, \mathrm{d}m(x) . \tag{3.22}$$

Calderon-Zygmund Inequality

Then for every $p \in (1, \infty)$ there exists a positive constant C_p that is independent of ϵ such that for every $\epsilon > 0$ the operator $\mathcal{K}_\epsilon$ satisfies the bound

$$\|\mathcal{K}_\epsilon u\|_{L^p} \leq C_p \|u\|_{L^p} \quad \text{for every } u \in L^p(\mathrm{d}m)\,, \tag{3.23}$$

Moreover, for every $u \in L^p(\mathrm{d}m)$ the limit

$$\mathcal{K}u = \lim_{\epsilon \to 0} \mathcal{K}_\epsilon u \quad \text{exists in } L^p(\mathrm{d}m)\,, \tag{3.24}$$

and the operator $\mathcal{K}$ so defined satisfies the bound

$$\|\mathcal{K}u\|_{L^p} \leq C_p \|u\|_{L^p} \quad \text{for every } u \in L^p(\mathrm{d}m)\,. \tag{3.25}$$

Summary of Convolution Inequalities

Our results regarding the convolution of two functions are summarized in the following table.

$$
\begin{array}{lll}
L^p * L^q \subset L^r & & \text{for } p, q, r \in [1, \infty] \text{ such that } \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}\,. \\
L^r_w * L^1 \subset L^r_w & & \text{for } r \in (1, \infty)\,. \\
L^p_w * L^q_w \subset L^r_w & & \text{for } p, q, r \in (1, \infty) \text{ such that } \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}\,. \\
L^p_w * L^q \subset L^r & & \text{for } p, q, r \in (1, \infty) \text{ such that } \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}\,. \\
CZ * L^r \subset L^r & & \text{for } r \in (1, \infty)\,.
\end{array}
$$

The first item follows from the Young convolution inequality, the second from the first Hardy-Littlewood-Sobolev inequality, the third from the second Hardy-Littlewood-Sobolev inequality, the fourth from the third Hardy-Littlewood-Sobolev inequality, and the last from the Calderon-Zygmund inequality, where CZ denotes all functions of the Calderon-Zygmund form (3.20).

Poisson Equation

For $\mathrm{D} > 2$ the Green function g of $-\Delta_x$ over $\mathbb{R}^{\mathrm{D}}$ is given by

$$g(x) = \frac{1}{|\mathbb{S}^{\mathrm{D}-1}|} |x|^{-\mathrm{D}+2} .$$

If u is the solution of the Poisson equation $-\Delta_x u = f$ for some $f \in L^p(\mathrm{d}m)$ then formally

$$u = g * f , \qquad \nabla_x u = (\nabla_x g) * f , \qquad \nabla_x^2 u = (\nabla_x^2 g) * f ,$$

where

$$\nabla_x g(x) = -\frac{\mathrm{D}-2}{|\mathbb{S}^{\mathrm{D}-1}|} |x|^{-\mathrm{D}+1} \frac{x}{|x|} , \quad \nabla_x^2 g(x) = \frac{\mathrm{D}-2}{|\mathbb{S}^{\mathrm{D}-1}|} |x|^{-\mathrm{D}} \left(\mathrm{D} \frac{x \otimes x}{|x|^2} - I \right) .$$

Poisson Equation

Because

$$|\nabla_x g(x)| = \frac{\mathrm{D}-2}{|\mathbb{S}^{\mathrm{D}-1}|}\,|x|^{-\mathrm{D}+1}\,, \qquad |\nabla_x^2 g(x)| = \frac{(\mathrm{D}-2)(\mathrm{D}-1)}{|\mathbb{S}^{\mathrm{D}-1}|}\,|x|^{-\mathrm{D}}\,,$$

we see that

$$g \in L_w^{\frac{\mathrm{D}}{\mathrm{D}-2}}(\mathrm{d}m)\,, \qquad \nabla_x g \in L_w^{\frac{\mathrm{D}}{\mathrm{D}-1}}(\mathrm{d}m)\,, \qquad \nabla_x^2 g \in CZ(\mathrm{d}m)\,,$$

where $CZ(\mathrm{d}m)$ denotes the set of all functions that have the Calderon-Zygmund form (3.20).

Poisson Equation

Hence, if $f \in L^p(\mathrm{d}m)$ then

$$
\begin{aligned}
u &\in L^{\frac{p\mathrm{D}}{\mathrm{D}-2p}}(\mathrm{d}m) && \text{when } p \in \left(1, \tfrac{\mathrm{D}}{2}\right), \\
\nabla_x u &\in L^{\frac{p\mathrm{D}}{\mathrm{D}-p}}(\mathrm{d}m) && \text{when } p \in (1, \mathrm{D}), \\
\nabla_x^2 u &\in L^p(\mathrm{d}m) && \text{when } p \in (1, \infty).
\end{aligned}
$$

The last result shows that solutions of the Poisson equation gain two derivatives.

Helmholtz Equation

The Green function g of $-\Delta_x + \kappa^2$ over $\mathbb{R}^3$ is given by

$$g(x) = \frac{1}{4\pi} \frac{e^{-\kappa|x|}}{|x|} .$$

If u is the solution of the Helmholtz equation $-\Delta_x u + \kappa^2 u = f$ for some $f \in L^p(\mathrm{d}m)$ then formally

$$u = g * f \, , \qquad \nabla_x u = (\nabla_x g) * f \, , \qquad \nabla_x^2 u = (\nabla_x^2 g) * f \, ,$$

where

$$\nabla_x g(x) = -\frac{1}{4\pi} \frac{e^{-\kappa|x|}}{|x|^2} \left(1 + \kappa|x|\right) \frac{x}{|x|} \, ,$$

$$\nabla_x^2 g(x) = \frac{1}{4\pi} \frac{e^{-\kappa|x|}}{|x|^3} \left(1 + \kappa|x|\right) \left(3 \frac{x \otimes x}{|x|^2} - I\right) + \frac{\kappa^2}{4\pi} \frac{e^{-\kappa|x|}}{|x|} \frac{x \otimes x}{|x|^2} \, .$$

Helmholtz Equation

Because

$$|\nabla_x g(x)| = \frac{1}{4\pi} \frac{e^{-\kappa|x|}}{|x|^2} (1 + \kappa|x|) \,,$$

we see that

$$\begin{aligned} g &\in L^q(\mathrm{d}m) \quad \text{for every } q \in [1,3) \qquad \text{and} \qquad & g &\in L^3_w(\mathrm{d}m) \,, \\ \nabla_x g &\in L^q(\mathrm{d}m) \quad \text{for every } q \in [1,\tfrac{3}{2}) \qquad \text{and} \qquad & \nabla_x g &\in L^{\frac{3}{2}}_w(\mathrm{d}m) \,. \end{aligned}$$

Helmholtz Equation

Hence, if $f \in L^p(\mathrm{d}m)$ then

$$
u \in L^r(\mathrm{d}m) \quad \begin{cases} \text{for every } r \in [p,\infty] & \text{when } p \in (\frac{3}{2},\infty)\,, \\ \text{for every } r \in [p,\infty) & \text{when } p = \frac{3}{2}\,, \\ \text{for every } r \in [p,\frac{3p}{3-2p}) & \text{when } p \in (1,\frac{3}{2})\,, \\ \text{for every } r \in [1,3) & \text{when } p = 1\,, \end{cases}
$$

$$
\nabla_x u \in L^r(\mathrm{d}m) \quad \begin{cases} \text{for every } r \in [p,\infty] & \text{when } p \in (3,\infty)\,, \\ \text{for every } r \in [p,\infty) & \text{when } p = 3\,, \\ \text{for every } r \in [p,\frac{3p}{3-p}) & \text{when } p \in (1,3)\,, \\ \text{for every } r \in [1,\frac{3}{2}) & \text{when } p = 1\,, \end{cases}
$$

In particular, we see that $u \in L^p(dm)$ and $\nabla_x u \in L^p(dm)$.

Helmholtz Equation

Finally, notice that $\nabla_x^2 g = H_1(x) + H_2(x)$ where H_1 and H_2 are the matrix-valued functions

$$H_1(x) = \frac{1}{4\pi} \frac{e^{-\kappa|x|}\,(1+\kappa|x|)}{|x|^3} \left(3\frac{x \otimes x}{|x|^2} - I \right), \quad H_2(x) = \frac{\kappa^2}{4\pi} \frac{e^{-\kappa|x|}}{|x|} \frac{x \otimes x}{|x|^2}.$$

Because

$$|H_1(x)| = \frac{1}{2\pi} \frac{e^{-\kappa|x|}\,(1+\kappa|x|)}{|x|^3}, \qquad |H_2(x)| = \frac{\kappa^2}{4\pi} \frac{e^{-\kappa|x|}}{|x|},$$

we see that $H_1 \in CZ(\mathrm{d}m)$ while

$$H_2 \in L^q(\mathrm{d}m) \quad \text{for every } q \in [1,3), \qquad \text{and} \qquad H_2 \in L^3_w(\mathrm{d}m).$$

Helmholtz Equation

In particular, we see that if $f \in L^p(\mathrm{d}m)$ then

$$\nabla_x^2 u = H_1 * f + H_2 * f \in L^p(\mathrm{d}m)\,, \qquad \text{when } p \in (1,\infty)\,.$$

Therefore, as with the Poisson equation, solutions of $-\Delta_x u + \kappa^2 u = f$ gain two derivatives.

General Integral Operators

We return to general linear integral operators $\mathcal{K}$ in the form

$$\mathcal{K}u(y) = \int k(x,y)\, u(x)\, \mathrm{d}\mu(x)\,, \tag{5.26}$$

where the kernel k is a complex-valued measurable function with respect to the σ-algebra $\Sigma_{\mu\otimes\nu}$.

Recall that $(X, \Sigma_\mu, \mathrm{d}\mu)$ and $(Y, \Sigma_\nu, \mathrm{d}\nu)$ are positive σ-finite measure spaces.

Recall too that $M(\mathrm{d}\mu)$ and $M(\mathrm{d}\nu)$ are the spaces of all complex-valued $\mathrm{d}\mu$-measurable and $\mathrm{d}\nu$-measurable functions respectively.

Adjoint Operator

Let $\mathcal{K}^*$ denote formal adjoint of $\mathcal{K}$, which is given by

$$\mathcal{K}^* v(x) = \int \overline{k(x,y)}\, v(y)\, \mathrm{d}\nu(y)\,. \tag{5.27}$$

The operator $\mathcal{K}$ is bounded from $L^p(\mathrm{d}\mu)$ to $L^{q^*}(\mathrm{d}\nu)$ if and only if $\mathcal{K}^*$ is bounded from $L^q(\mathrm{d}\nu)$ to $L^{p^*}(\mathrm{d}\mu)$ where $p^*, q \in [1,\infty]$ are determined by the duality relations

$$\frac{1}{p} + \frac{1}{p^*} = 1\,, \quad \text{and} \quad \frac{1}{q} + \frac{1}{q^*} = 1\,. \tag{5.28}$$

Moreover, $\|\mathcal{K}^*\|_{B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))} = \|\mathcal{K}\|_{B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))}$.

Paired Bounds

Lemma

Let $k \in M(\mathrm{d}\mu\,\mathrm{d}\nu)$ *and* $C \in [0,\infty)$ *such that for every* $u \in L^p(\mathrm{d}\mu)$ *and* $v \in L^q(\mathrm{d}\nu)$ *we have*

$$\iint \left|k(x,y)\,u(x)\,\overline{v(y)}\right| \mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \leq C\,\|u\|_{L^p}\|v\|_{L^q}\,. \tag{5.29}$$

Then $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))$ *and* $\mathcal{K}^* \in B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ *with*

$$\|\mathcal{K}\|_{B(L^p,L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q,L^{p^*})} \leq C\,. \tag{5.30}$$

Remark. The measures $\mathrm{d}\mu$ and $\mathrm{d}\nu$ will be dropped from the notation for norms when there is no confusion about what measures are involved.

Iterated Norm Bounds

Lemma

Let $p, q \in [1, \infty]$. *Let the kernel* k *satisfy the bound*

$$\|k\|_{L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))} = \left(\left(\int |k(x,y)|^{p^*} \mathrm{d}\mu(x) \right)^{\frac{q^*}{p^*}} \mathrm{d}\nu(y) \right)^{\frac{1}{q^*}} < \infty \,. \tag{5.31}$$

Then for every $u \in L^p(\mathrm{d}\mu)$ *and* $v \in L^q(\mathrm{d}\nu)$ *we have*

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^{q^*}(L^{p^*})} \, \|u\|_{L^p} \, \|v\|_{L^q} \,. \tag{5.32}$$

Iterated Norm Bounds

Lemma

Similarly, let the kernel k satisfy the bound

$$\|k\|_{L^{p^*}(\mathrm{d}\mu;L^{q^*}(\mathrm{d}\nu))} = \left(\left(\int |k(x,y)|^{q^*}\,\mathrm{d}\nu(y)\right)^{\frac{p^*}{q^*}}\mathrm{d}\mu(x)\right)^{\frac{1}{p^*}} < \infty\,. \qquad (5.33)$$

Then for every $u \in L^p(\mathrm{d}\mu)$ and $v \in L^q(\mathrm{d}\nu)$ we have

$$\iint \left|k(x,y)\,u(x)\,\overline{v(y)}\right|\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \leq \|k\|_{L^{p^*}(L^{q^*})}\,\|u\|_{L^p}\,\|v\|_{L^q}\,. \qquad (5.34)$$

Iterated Norm Bounds

Remark. The spaces $L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))$ and $L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))$ are called iterated spaces. They are equipped with the so-called iterated norms $\|\cdot\|_{L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))}$ and $\|\cdot\|_{L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))}$ defined above by (5.31) and (5.33). The bounds (5.32) and (5.34) are called iterated norm bounds.

Remark. The Minkowski inequality for integrals implies that

$$\begin{aligned} \|k\|_{L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))} &\leq \|k\|_{L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))} \quad \text{whenever } p^* \leq q^* \,, \\ \|k\|_{L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))} &\leq \|k\|_{L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))} \quad \text{whenever } q^* \leq p^* \,. \end{aligned} \tag{5.35}$$

Iterated Norm Bounds

In the first case we can conclude that the first iterated norm bound (5.32) is the sharper one, whereby we conclude by Lemma 6 that $\mathcal{K} \in B(L^p, L^{q^*})$ and $\mathcal{K}^* \in B(L^q, L^{p^*})$ with

$$\|\mathcal{K}\|_{B(L^p,L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q,L^{p^*})} \leq \|k\|_{L^{q^*}(L^{p^*})} \quad \text{for every } k \in L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu)) \,. \tag{5.36}$$

In the second case we can conclude that the second iterated norm bound (5.34) is the sharper one, whereby we conclude by Lemma 6 that $\mathcal{K} \in B(L^p, L^{q^*})$ and $\mathcal{K}^* \in B(L^q, L^{p^*})$ with

$$\|\mathcal{K}\|_{B(L^p,L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q,L^{p^*})} \leq \|k\|_{L^{p^*}(L^{q^*})} \quad \text{for every } k \in L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu)) \,. \tag{5.37}$$

Iterated Norm Bounds

Remark: When either $1 \leq p^* \leq q^* < \infty$ and $k \in L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))$ or $1 \leq q^* \leq p^* < \infty$ and $k \in L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))$ then we can conclude that the bounded operators $\mathcal{K}$ and $\mathcal{K}^*$ from (5.36) and (5.37) are moreover compact. This is because one can show that the finite-rank kernels are dense in the spaces $L^{q^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))$ and $L^{p^*}(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu))$. The classical Hilbert-Schmidt compactness criterion is the special case $p = q = 2$.

Iterated Norm Bounds

Remark: When $p = q$ in the iterated spaces $L^{p^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))$ and $L^{p^*}(\mathrm{d}\mu; L^{p^*}(\mathrm{d}\nu))$ coincide with

$$L^{p^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu)) = L^{p^*}(\mathrm{d}\mu; L^{p^*}(\mathrm{d}\nu)) = L^{p^*}(\mathrm{d}\mu\, \mathrm{d}\nu)\,.$$

Moreover, the iterated norms given by (5.31) and (5.33) also coincide with

$$\|k\|_{L^{p^*}(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu))} = \|k\|_{L^{p^*}(\mathrm{d}\mu; L^{p^*}(\mathrm{d}\nu))} = \|k\|_{L^{p^*}(\mathrm{d}\mu\, \mathrm{d}\nu)}\,.$$

If these are finite then $\mathcal{K}$ is bounded from $L^p(\mathrm{d}\mu)$ to $L^{p^*}(\mathrm{d}\nu)$ and $\mathcal{K}^*$ is bounded from $L^p(\mathrm{d}\nu)$ to $L^{p^*}(\mathrm{d}\mu)$. If moreover $p^* < \infty$ then $\mathcal{K}$ and $\mathcal{K}^*$ are also compact by the previous remark.

Iterated Norm Bounds

Remark: In some cases the iterated norm bounds (5.32) and (5.34) are sharp. Specifically, it can be shown that when $p \in [1, \infty]$ and $q = 1$ one has

$$\|\mathcal{K}\|_{B(L^p, L^\infty)} = \|\mathcal{K}^*\|_{B(L^1, L^{p^*})} = \|k\|_{L^\infty(L^{p^*})} \quad \text{for every } k \in L^\infty(\mathrm{d}\nu; L^{p^*}(\mathrm{d}\mu)) ,$$

while when $p = 1$ and $q \in [1, \infty]$ one has

$$\|\mathcal{K}\|_{B(L^1, L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q, L^\infty)} = \|k\|_{L^\infty(L^{q^*})} \quad \text{for every } k \in L^\infty(\mathrm{d}\mu; L^{q^*}(\mathrm{d}\nu)) .$$

Young Integral Operator Bound

The results of the previous section include the following. If $k \in L^\infty(\mathrm{d}\mu\,\mathrm{d}\nu)$ then for every $u \in L^1(\mathrm{d}\mu)$ and $v \in L^1(\mathrm{d}\nu)$ we have

$$\iint \left|k(x,y)\,u(x)\,\overline{v(y)}\right| \mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \leq \|k\|_{L^\infty(\mathrm{d}\mu\,\mathrm{d}\nu)} \|u\|_{L^1}\,\|v\|_{L^1}\,. \tag{5.38}$$

If $k \in L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))$ for some $r \in [1,\infty)$ then for every $u \in L^1(\mathrm{d}\mu)$ and $v \in L^{r^*}(\mathrm{d}\nu)$ we have

$$\iint \left|k(x,y)\,u(x)\,\overline{v(y)}\right| \mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \leq \|k\|_{L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))} \|u\|_{L^1}\,\|v\|_{L^{r^*}}\,. \tag{5.39}$$

If $k \in L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu))$ for some $r \in [1,\infty)$ then for every $u \in L^{r^*}(\mathrm{d}\mu)$ and $v \in L^1(\mathrm{d}\nu)$ we have

$$\iint \left|k(x,y)\,u(x)\,\overline{v(y)}\right| \mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \leq \|k\|_{L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu))} \|u\|_{L^{r^*}}\,\|v\|_{L^1}\,. \tag{5.40}$$

Young Integral Operator Bound

If $k \in L^\infty(L^r)(\mathrm{d}\mu, \mathrm{d}\nu) = L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu)) \cap L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))$ for some $r \in [1, \infty)$ then, in addition to the bounds (5.39) and (5.40), we have an entire family of Young integral operator bounds.

Theorem

Let $k \in L^\infty(L^r)(\mathrm{d}\mu, \mathrm{d}\nu)$ for some $r \in [1, \infty)$. Let $p, q \in [1, r^*]$ satisfy the relation

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2\,. \tag{5.41}$$

Then for every $u \in L^p(\mathrm{d}\mu)$ and $v \in L^q(\mathrm{d}\nu)$ we have

$$\begin{aligned} &\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \\ &\qquad \leq \|k\|_{L^\infty(\mathrm{d}\mu; L^r(\mathrm{d}\nu))}^{\frac{r}{p^*}} \|k\|_{L^\infty(\mathrm{d}\nu; L^r(\mathrm{d}\mu))}^{\frac{r}{q^*}} \|u\|_{L^p} \|v\|_{L^q}\,. \end{aligned} \tag{5.42}$$

Young Integral Operator Bound

Theorem

Moreover, we have $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))$ *and* $\mathcal{K}^* \in B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ *with*

$$\|\mathcal{K}\|_{B(L^p,L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q,L^{p^*})} \leq \|k\|_{L^\infty(\mathrm{d}\mu;L^r(\mathrm{d}\nu))}^{\frac{r}{p^*}} \|k\|_{L^\infty(\mathrm{d}\nu;L^r(\mathrm{d}\mu))}^{\frac{r}{q^*}} . \quad (5.43)$$

Remark. The case $r = \infty$ is already covered by (5.38) because in that case relation (5.41) would require that $p = q = 1$. The case $r \in [1, \infty)$ and $p = 1$ is already covered by (5.39) because in that case relation (5.41) would require that $q = r^*$. The case $r \in [1, \infty)$ and $q = 1$ is already covered by (5.40) because in that case relation (5.41) would require that $p = r^*$.

Interpolation Bounds

The family of Young integral bounds (5.42) belongs to the larger class of interpolation bounds. We will give interpolation bounds in the following setting. Suppose that for some $p_0, q_0, p_1, q_1 \in [1, \infty]$ the kernel k satisfies the bounds

$$\begin{aligned} &\iint \left|k(x,y)\, u(x)\, \overline{v(y)}\right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq C_0\, \|u\|_{L^{p_0}} \|v\|_{L^{q_0}} \\ &\qquad \text{for every } u \in L^{p_0}(\mathrm{d}\mu) \text{ and } v \in L^{q_0}(\mathrm{d}\nu)\,, \\ &\iint \left|k(x,y)\, u(x)\, \overline{v(y)}\right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq C_1\, \|u\|_{L^{p_1}} \|v\|_{L^{q_1}} \\ &\qquad \text{for every } u \in L^{p_1}(\mathrm{d}\mu) \text{ and } v \in L^{q_1}(\mathrm{d}\nu)\,, \end{aligned} \tag{6.44}$$

These bounds imply the operator $\mathcal{K}$ belongs to $B(L^{p_0}(\mathrm{d}\mu), L^{q_0^*}(\mathrm{d}\nu))$ and to $B(L^{p_1}(\mathrm{d}\mu), L^{q_1^*}(\mathrm{d}\nu))$, where the usual duality relations $\frac{1}{q_0} + \frac{1}{q_0^*} = 1$ and $\frac{1}{q_1} + \frac{1}{q_1^*} = 1$ hold. Interpolation will allow us to extend all of these results to other spaces.

Interpolation Bounds

Lemma

If the kernel k satisfies the bounds (6.44) for some $p_0, q_0, p_1, q_1 \in [1, \infty]$ *then for every* $t \in [1, \infty]$ *it satisfies the interpolation bound*

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq C_0^{\frac{1}{t^*}}\, C_1^{\frac{1}{t}}\, \|u\|_{L^p} \|v\|_{L^q} \quad \text{for every } u \in L^p(\mathrm{d}\mu) \text{ and } v \in L^q(\mathrm{d}\nu)\,, \tag{6.45}$$

where $t^* \in [1, \infty]$ *satisfies* $\frac{1}{t} + \frac{1}{t^*} = 1$*, and* p*,* $q \in [1, \infty]$ *satisfy the interpolation relations*

$$\frac{1}{p} = \frac{1}{t^* p_0} + \frac{1}{t p_1}\,, \qquad \frac{1}{q} = \frac{1}{t^* q_0} + \frac{1}{t q_1}\,. \tag{6.46}$$

Interpolation Bounds

Lemma

Moreover, we have $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))$ *and* $\mathcal{K}^* \in B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ *with*

$$\|\mathcal{K}\|_{B(L^p,L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q,L^{p^*})} \leq C_0^{\frac{1}{t^*}} C_1^{\frac{1}{t}} . \tag{6.47}$$

Here the usual duality relations $\frac{1}{p} + \frac{1}{p^*} = 1$, *and* $\frac{1}{q} + \frac{1}{q^*} = 1$ *hold.*

Interpolation Bounds

We now apply the Interpolation Lemma 11 to a kernel k which for some $r, s \in [1, \infty]$ satisfies the bounds

$$\begin{aligned}
\|k\|_{L^s(\mathrm{d}\nu;L^r(\mathrm{d}\mu))} &= \left(\int \left(\int |k(x,y)|^r \mathrm{d}\mu(x) \right)^{\frac{s}{r}} \mathrm{d}\nu(y) \right)^{\frac{1}{s}} < \infty \,, \\
\|k\|_{L^s(\mathrm{d}\mu;L^r(\mathrm{d}\nu))} &= \left(\int \left(\int |k(x,y)|^r \mathrm{d}\nu(y) \right)^{\frac{s}{r}} \mathrm{d}\mu(x) \right)^{\frac{1}{s}} < \infty \,.
\end{aligned} \tag{6.48}$$

Without loss of generality we can assume $r \leq s$ because in that case $\|k\|_{L^s(L^r)} \leq \|k\|_{L^r(L^s)}$ for each of the above norms. We can assume moreover that $r < s$ because when $r = s$ the bounds in (6.48) coincide, so the Interpolation Lemma cannot yield further boundedness results.

Interpolation Bounds

The iterated norm bounds show that the bounds (6.48) on k imply the following. Because $k \in L^s(\mathrm{d}\nu; L^r(\mathrm{d}\mu))$, then for every $u \in L^{r^*}(\mathrm{d}\mu)$ and $v \in L^{s^*}(\mathrm{d}\nu)$ we have

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^s(\mathrm{d}\nu;(L^r(\mathrm{d}\mu))} \|u\|_{L^{r^*}} \|v\|_{L^{s^*}} . \tag{6.49}$$

Because $k \in L^s(\mathrm{d}\mu; L^r(\mathrm{d}\nu))$, then for every $u \in L^{s^*}(\mathrm{d}\mu)$ and $v \in L^{r^*}(\mathrm{d}\nu)$ we have

$$\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \leq \|k\|_{L^s(\mathrm{d}\mu;(L^r(\mathrm{d}\nu))} \|u\|_{L^{s^*}} \|v\|_{L^{r^*}} . \tag{6.50}$$

In this section we show that because $k \in L^s(L^r)(\mathrm{d}\mu, \mathrm{d}\nu) = L^s(\mathrm{d}\nu; L^r(\mathrm{d}\mu)) \cap L^s(\mathrm{d}\mu; L^r(\mathrm{d}\nu))$ then, in addition to the bounds (6.49) and (6.50), we have a family of interpolation bounds.

Interpolation Bounds

Theorem

Let $k \in L^s(L^r)(\mathrm{d}\mu\,\mathrm{d}\nu)$ *for some* $r, s \in [1, \infty]$ *such that* $r < s$. *Let* $p, q \in [s^*, r^*]$ *satisfy the relation*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 2\,. \tag{6.51}$$

Then for every $u \in L^p(\mathrm{d}\mu)$ *and* $v \in L^q(\mathrm{d}\nu)$ *we have the interpolation bound*

$$\begin{aligned} &\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \\ &\qquad \leq \|k\|_{L^s(\mathrm{d}\nu; L^r(\mathrm{d}\mu))}^{\frac{1}{t^*}} \|k\|_{L^s(\mathrm{d}\mu; L^r(\mathrm{d}\nu))}^{\frac{1}{t}} \|u\|_{L^p} \|v\|_{L^q}\,, \end{aligned} \tag{6.52}$$

Interpolation Bounds

Theorem

where t^* *and* t *are given by*

$$\frac{1}{t^*} = \frac{\dfrac{1}{s^*} - \dfrac{1}{p}}{\dfrac{1}{s^*} - \dfrac{1}{r^*}} = \frac{\dfrac{1}{q} - \dfrac{1}{r^*}}{\dfrac{1}{s^*} - \dfrac{1}{r^*}}, \qquad \frac{1}{t} = \frac{\dfrac{1}{p} - \dfrac{1}{r^*}}{\dfrac{1}{s^*} - \dfrac{1}{r^*}} = \frac{\dfrac{1}{s^*} - \dfrac{1}{q}}{\dfrac{1}{s^*} - \dfrac{1}{r^*}}. \tag{6.53}$$

Moreover, we have $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{q^*}(\mathrm{d}\nu))$ *and* $\mathcal{K}^* \in B(L^q(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ *with*

$$\|\mathcal{K}\|_{B(L^p, L^{q^*})} = \|\mathcal{K}^*\|_{B(L^q, L^{p^*})} \leq \|k\|_{L^s(\mathrm{d}\nu; L^r(\mathrm{d}\mu))}^{\frac{1}{t^*}} \|k\|_{L^s(\mathrm{d}\mu; L^r(\mathrm{d}\nu))}^{\frac{1}{t}}. \tag{6.54}$$

When $s < \infty$ *the operators* $\mathcal{K}$ *and* $\mathcal{K}^*$ *are also compact.*

Interpolation Bounds

Remark: When $s = \infty$ this reduces to the Young Integral Operator Theorem.

Remark: When $r \in [1,2]$ (so that $r \leq r^*$) and $s = r^*$ then for every $p \in [r, r^*]$ one sees that $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^p(\mathrm{d}\nu))$ and $\mathcal{K}^* \in B(L^{p^*}(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ with

$$\|\mathcal{K}\|_{B(L^p,L^p)} = \|\mathcal{K}^*\|_{B(L^{p^*},L^{p^*})} \leq \|k\|_{L^{r^*}(\mathrm{d}\nu;L^r(\mathrm{d}\mu))}^{\frac{1}{t^*}} \|k\|_{L^{r^*}(\mathrm{d}\mu;L^r(\mathrm{d}\nu))}^{\frac{1}{t}},$$

where t is given by (6.53). In this case $q = p^*$.

Remark: Let p satisfy $\frac{2}{p} = \frac{1}{r^*} + \frac{1}{s^*}$. Notice that p is the harmonic mean of r^* and s^*, so that $p \in [s^*, r^*]$. One sees that $\mathcal{K} \in B(L^p(\mathrm{d}\mu), L^{p^*}(\mathrm{d}\nu))$ and $\mathcal{K}^* \in B(L^p(\mathrm{d}\nu), L^{p^*}(\mathrm{d}\mu))$ with

$$\|\mathcal{K}\|_{B(L^p,L^{p^*})} = \|\mathcal{K}^*\|_{B(L^p,L^{p^*})} \leq \|k\|_{L^s(\mathrm{d}\nu;L^r(\mathrm{d}\mu))}^{\frac{1}{2}} \|k\|_{L^s(\mathrm{d}\mu;L^r(\mathrm{d}\nu))}^{\frac{1}{2}}.$$

In this case $q = p$.

First Hardy-Littlewood Inequality

Theorem

Let $r \in (1, \infty)$. For every kernel k that satisfies

$$\|k\|_{L^\infty(\mathrm{d}\mu; L^r_w(\mathrm{d}\nu))} = \operatorname*{ess\,sup}_{x \in X} \left\{ \sup_{E \in \Sigma_\nu} \left\{ \frac{1}{\nu(E)^{\frac{1}{r^*}}} \int_E |k(x, y)| \,\mathrm{d}\nu(y) \ : \ \nu(E) \in (0, \infty) \right\} \right\} \tag{7.55}$$

the integral operator $\mathcal{K}$ defined by (5.26) satisfies the bound

$$\|\mathcal{K}u\|_{L^r_w(\mathrm{d}\nu)} \leq \|k\|_{L^\infty(\mathrm{d}\mu; L^r_w(\mathrm{d}\nu))} \, \|u\|_{L^1(\mathrm{d}\mu)} \quad \text{for every } u \in L^1(\mathrm{d}\mu) \,. \tag{7.56}$$

Remark. Bound (7.56) shows that the operator $\mathcal{K}$ is bounded from $L^1(\mathrm{d}\mu)$ into $L^r_w(\mathrm{d}\nu)$ with

$$\|\mathcal{K}\|_{B(L^1, L^r_w)} \leq \|k\|_{L^\infty(L^r_w)} \quad \text{for every } k \in L^\infty(\mathrm{d}\mu; L^r_w(\mathrm{d}\nu)) \,. \tag{7.57}$$

This result should be compared to (5.37) with $p = 1$ and $q = r^*$.

Second Hardy-Littlewood Inequality

Theorem

Let $p, q, r \in (1, \infty)$ *satisfy the relation*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2\,. \tag{7.58}$$

Let the kernel k *satisfy*

$$\|k\|_{L^\infty(\mathrm{d}\nu;L^r_w(\mathrm{d}\mu))} = \operatorname*{ess\,sup}_{y\in Y}\left\{ \sup_{E\in\Sigma_\mu} \left\{ \frac{1}{\mu(E)^{\frac{1}{r^*}}} \int_E |k(x,y)|\,\mathrm{d}\mu(x) \;:\; \mu(E) \in (0, \infty\right.\right.$$
$$< \infty\,,$$
$$\|k\|_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))} = \operatorname*{ess\,sup}_{x\in X}\left\{ \sup_{E\in\Sigma_\nu} \left\{ \frac{1}{\nu(E)^{\frac{1}{r^*}}} \int_E |k(x,y)|\,\mathrm{d}\nu(y) \;:\; \nu(E) \in (0, \infty\right.\right.$$
$$< \infty\,.$$

Second Hardy-Littlewood Inequality

Theorem

Then there exists a positive $C_w^{p,q,r}$ such that the integral operator $\mathcal{K}$ defined by (5.26) satisfies the bound

$$\|\mathcal{K}u\|_{L_w^{q^*}} \leq C_w^{p,q,r} \, \|k\|_{L^\infty(\mathrm{d}\nu; L_w^r(\mathrm{d}\mu))}^{\frac{r}{p^*}} \, \|k\|_{L^\infty(\mathrm{d}\mu; L_w^r(\mathrm{d}\nu))}^{\frac{r}{q^*}} \, [u]_{L_w^p} \qquad (7.60)$$

for every $k \in L^\infty(L_w^r)(\mathrm{d}\mu, \mathrm{d}\nu)$ and $u \in L_w^p(\mathrm{d}\mu)$.

Remark. We can establish (7.60) with

$$C_w^{p,q,r} = \frac{p^* q^* r^*}{p \; r} = p^* r^* + q^* \,. \qquad (7.61)$$

This $C_w^{p,q,r}$ is universal in the sense that it is independent of the underlying measure spaces $(X, \Sigma_\mu, \mathrm{d}\mu)$ and $(Y, \Sigma_\nu, \mathrm{d}\nu)$.

Third Hardy-Littlewood Inequality

Theorem

Let $p, q, r \in (1, \infty)$ *satisfy the relation*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2\,. \tag{7.62}$$

Let $k \in L^\infty(L^r_w)(\mathrm{d}\mu, \mathrm{d}\nu)$. *Then there exists a positive* $C^{p,q,r}$ *such that for every* $u \in L^p(\mathrm{d}\mu)$ *and* $v \in L^q(\mathrm{d}\nu)$

$$\begin{aligned} &\iint \left| k(x,y)\, u(x)\, \overline{v(y)} \right| \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \\ &\qquad \leq C^{p,q,r} \, \|k\|_{L^\infty(\mathrm{d}\nu; L^r_w(\mathrm{d}\mu))}^{\frac{r}{p^*}} \, \|k\|_{L^\infty(\mathrm{d}\mu; L^r_w(\mathrm{d}\nu))}^{\frac{r}{q^*}} \, \|u\|_{L^p} \, \|v\|_{L^q}\,. \end{aligned} \tag{7.63}$$

Third Hardy-Littlewood Inequality

Remark. We can establish (7.63) with

$$C^{p,q,r} = \frac{r^*}{pq}\left(\frac{p^*}{r}\right)^{\frac{1}{r}+\frac{r}{p^*r^*}}\left(\frac{q^*}{r}\right)^{\frac{1}{r}+\frac{r}{q^*r^*}} \leq \frac{p^*\,q^*\,r^*}{p\;q\;r^2}\,. \tag{7.64}$$

This $C^{p,q,r}$ is universal in the sense that it is independent of the underlying measure spaces $(X, \Sigma_\mu, \mathrm{d}\mu)$ and $(Y, \Sigma_\nu, \mathrm{d}\nu)$.

Third Hardy-Littlewood Inequality

Remark. Bound (7.63) shows that for every $k \in L^\infty(L^r_w)(\mathrm{d}\mu, \mathrm{d}\nu)$ the operator $\mathcal{K}$ defined by (5.26) is bounded from $L^p(\mathrm{d}\mu)$ into $L^{q^*}(\mathrm{d}\nu)$ with

$$\|\mathcal{K}\|_{B(L^p,L^{q^*})} \leq C^{p,q,r} \|k\|_{L^\infty(\mathrm{d}\nu;L^r_w(\mathrm{d}\mu))}^{\frac{r}{p^*}} \|k\|_{L^\infty(\mathrm{d}\mu;L^r_w(\mathrm{d}\nu))}^{\frac{r}{q^*}} . \tag{7.65}$$

Remark. This should be compared with bound (5.43) obtained from the Young integral operator bound (5.42). For each $r \in (1, \infty)$ that bound requires the kernel k to be in the more restrictive class $L^\infty(L^r)(\mathrm{d}\mu, \mathrm{d}\nu)$, but includes the cases $p = 1$ or $q = 1$. From (7.62) and (7.64) we see that $C^{p,q,r} \to \infty$ as either $(p, q^*) \to (1, r)$ or $(p, q^*) \to (r^*, \infty)$, whereby bound (7.65) breaks down in these limits. The breakdown at $(1, r)$ should be contrasted with bound (7.57), in which the range of $\mathcal{K}$ is $L^r_w(\mathrm{d}\nu)$ rather than $L^r(\mathrm{d}\nu)$.