**First In-Class Exam**
**Math 410, Professor David Levermore**
**due 4:30pm Tuesday, 6 October 2020**

**Indicate your answer to each part of each question clearly. Work that you do not want considered should be crossed out. Your reasoning must be given for full credit.** Good luck!

1. [10] Prove for every $n \in \mathbb{N}$ and every $a \in \mathbb{R}_+$ that
$$(1+a)^n \geq 1 + n\,a + \frac{n(n-1)}{2}\,a^2 + \frac{n(n-1)(n-2)}{6}\,a^3\,.$$

2. [15] Let $c_k = \sin(2k)$ for every $k \in \mathbb{Z}_+$.
   a. [10] Show that $c_k > .5$ frequently but not eventually.
   b. [5] Show that $\{c_k\}_{k\in\mathbb{Z}_+}$ is divergent.

3. [10] Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Show that if $\{a_k\}$ is convergent with
$$\lim_{k\to\infty} a_k = a > 0\,,$$
then
$$\limsup_{k\to\infty} a_k b_k = a \limsup_{k\to\infty} b_k\,.$$

4. [15] Let $a_0 > 0$ and define $\{a_k\}_{k\in\mathbb{N}}$ by $a_{k+1} = \frac{2}{3}(a_k + 1/a_k^2)$ for every $k \in \mathbb{N}$.
   a. [10] Prove that $\{a_k\}_{k\in\mathbb{N}}$ converges.
   b. [5] Evaluate $\lim_{k\to\infty} a_k$. Give your reasoning.

5. [10] Let $A, B \subset \mathbb{R}$. Show that $(A \cup B)^c = A^c \cup B^c$.
(Here $S^c$ denotes the closure of any $S \subset \mathbb{R}$.)

6. [10] Determine all $a \in \mathbb{R}$ for which
$$\sum_{k=0}^{\infty} \left(\frac{\sqrt{k^2+1}}{k^3+4}\right)^a \quad \text{converges}\,.$$
Give your reasoning.

7. [15] Determine the set $C \subset \mathbb{R}$ defined by
$$C = \left\{ x \in \mathbb{R} \,:\, \sum_{k=1}^{\infty} \frac{x^k}{k} \quad \text{converges} \right\}.$$
Give your reasoning. (The set is an interval. Be sure to check its endpoints!)

**More on the next page.**

8. [15] Use facts from calculus to prove the following.

a. [5] Show for every $x < 1$ and every $n \in \mathbb{Z}_+$ that
$$\sum_{k=1}^{n} \frac{x^k}{k} = \log\left(\frac{1}{1-x}\right) - \int_0^x \frac{y^n}{1-y}\, \mathrm{d}y\,.$$

b. [5] Show for every $n \in \mathbb{Z}_+$ that
$$\left|\int_0^x \frac{y^n}{1-y}\, \mathrm{d}y\right| \leq \begin{cases} \dfrac{1}{n+1}\dfrac{x^{n+1}}{1-x} & \text{for } x \in (0,1)\,, \\ \dfrac{1}{n+1}|x|^{n+1} & \text{for } x \leq 0\,. \end{cases}$$

c. [5] Let $C \subset \mathbb{R}$ be defined as in Problem 7. Show for every $x \in C$ that
$$\sum_{k=1}^{\infty} \frac{x^k}{k} = \log\left(\frac{1}{1-x}\right)\,.$$