First Exam Solutions
Math 410, Professor David Levermore
due 4:30pm Tuesday, 6 October 2020

1. [10] Prove for every $n \in \mathbb{N}$ and every $a \in \mathbb{R}_+$ that

$$(1+a)^n \geq 1 + n\,a + \frac{n(n-1)}{2}\,a^2 + \frac{n(n-1)(n-2)}{6}\,a^3 .$$

Solution. Let $a \in \mathbb{R}_+$. For $n = 0$, 1, 2, and 3 we have

$$\begin{aligned}(1+a)^0 &= 1\,,\\ (1+a)^1 &= 1 + a\,,\\ (1+a)^2 &= 1 + 2a + a^2\,,\\ (1+a)^3 &= 1 + 3a + 3a^2 + a^3\,,\end{aligned}$$

which shows that the desired inequality holds as an equality for $n = 0$, 1, 2, and 3. For $n \geq 4$ the binomial formula yields

$$\begin{aligned}(1+a)^n &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k\\ &= 1 + n\,a + \frac{n(n-1)}{2}\,a^2 + \frac{n(n-1)(n-2)}{6}\,a^3 + \sum_{k=4}^{n} \frac{n!}{k!(n-k)!} a^k\\ &> 1 + n\,a + \frac{n(n-1)}{2}\,a^2 + \frac{n(n-1)(n-2)}{6}\,a^3\,,\end{aligned}$$

which shows that the desired inequality holds as a stricty inequality for $n \geq 4$. $\square$

Alternative Solution. Let $a \in \mathbb{R}_+$. For $n = 0$ we have

$$(1+a)^0 = 1\,,$$

which shows that the desired inequality holds when $n = 0$. Now suppose that it holds for some $n \in \mathbb{N}$. We then have

$$\begin{aligned}(1+a)^{n+1} &= (1+a)(1+a)^n\\ &\geq (1+a)\Big(1 + n\,a + \frac{n(n-1)}{2}\,a^2 + \frac{n(n-1)(n-2)}{6}\,a^3\Big)\\ &= 1 + (n+1)\,a + \left(\frac{n(n-1)}{2} + n\right) a^2\\ &\qquad + \left(\frac{n(n-1)(n-2)}{6} + \frac{n(n-1)}{2}\right) a^3 + \frac{n(n-1)(n-2)}{6}\,a^4\\ &= 1 + (n+1)\,a + \frac{n(n+1)}{2}\,a^2 + \frac{n(n-1)(n+1)}{6}\,a^3 + \frac{n(n-1)(n-2)}{6}\,a^4\\ &\geq 1 + (n+1)\,a + \frac{(n+1)n}{2}\,a^2 + \frac{(n+1)n(n-1)}{6}\,a^3\,,\end{aligned}$$

whereby the desired inequality holds for $n+1$. By induction the desired inequality holds for every $n \in \mathbb{N}$. $\square$

2. [15] Let $c_k = \sin(2k)$ for every $k \in \mathbb{Z}_+$.
   a. [10] Show that $c_k > .5$ frequently but not eventually.
   b. [5] Show that $\{c_k\}_{k\in\mathbb{Z}_+}$ is divergent.

Solution (a). Define the sets

$$S = \big\{x \in \mathbb{R} \,:\, \sin(2x) > .5\big\}\,, \qquad S' = \big\{x \in \mathbb{R} \,:\, \sin(2x) < -.5\big\}\,.$$

These sets are unions of disjoint open intervals given by

$$S = \bigcup_{n\in\mathbb{Z}} \left(n\pi + \frac{\pi}{12}\,,\, n\pi + \frac{5\pi}{12}\right)\,, \qquad S' = \bigcup_{n\in\mathbb{Z}} \left(n\pi + \frac{7\pi}{12}\,,\, n\pi + \frac{11\pi}{12}\right)\,.$$

The length of each interval in these unions is $\frac{\pi}{3}$. Because these lengths are greater than 1, every interval in these unions contains at least one integer. For each $n \in \mathbb{N}$ let $k_n \in \mathbb{Z}_+$ and $k_n' \in \mathbb{Z}_+$ such that

$$k_n \in \left(n\pi + \frac{\pi}{12}\,,\, n\pi + \frac{5\pi}{12}\right) \qquad k_n' \in \left(n\pi + \frac{7\pi}{12}\,,\, n\pi + \frac{11\pi}{12}\right)\,.$$

Because $\{c_{k_n}\}_{n\in\mathbb{N}}$ is a subsequence of $\{c_k\}_{k\in\mathbb{Z}_+}$ contained in S we see that $\{c_k\}_{k\in\mathbb{Z}_+}$ is in S freguently. Because $\{c_{k_n'}\}_{n\in\mathbb{N}}$ is a subsequence of $\{c_k\}_{k\in\mathbb{Z}_+}$ contained in S' we see that $\{c_k\}_{k\in\mathbb{Z}_+}$ is in S' freguently.

Solution (b). The sequence $\{c_k\}_{k\in\mathbb{Z}_+}$ diverges because $c_k > .5$ frequently and $c_k < -.5$ frequently.

3. [10] Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Show that if $\{a_k\}$ is convergent with

$$\lim_{k\to\infty} a_k = a > 0\,,$$

then

$$\limsup_{k\to\infty} a_k b_k = a \limsup_{k\to\infty} b_k\,.$$

Solution. Three cases must be considered:

$$\limsup_{k\to\infty} b_k = -\infty\,, \qquad \limsup_{k\to\infty} b_k = \infty\,, \qquad \limsup_{k\to\infty} b_k = b \quad \text{for some } b \in \mathbb{R}\,.$$

First, suppose that

$$\limsup_{k\to\infty} b_k = -\infty\,.$$

We must show that

$$\limsup_{k\to\infty} a_k b_k = -\infty\,.$$

Let $m \in \mathbb{R}_-$. Because $a > 0$ our supposition implies that

$$b_k < \frac{2m}{a} < 0 \quad \text{eventually}\,.$$

Because $a_k \to a$ as $k \to \infty$ and $a > 0$ we know that

$$0 < \frac{a}{2} < a_k \quad \text{eventually}\,.$$

Hence, for every $m \in \mathbb{R}_-$ we have

$$a_k \cdot b_k < \frac{a}{2} \cdot \frac{2m}{a} = m \quad \text{eventually}\,.$$

Therefore

$$\limsup_{k\to\infty} a_k b_k = -\infty\,.$$

Next, suppose that

$$\limsup_{k\to\infty} b_k = \infty\,.$$

We must show that

$$\limsup_{k\to\infty} a_k b_k = \infty\,.$$

Let $m \in \mathbb{R}_+$. Because $a > 0$ our supposition implies that

$$b_k > \frac{2m}{a} > 0 \quad \text{eventually}\,.$$

Because $a_k \to a$ as $k \to \infty$ and $a > 0$ we know that

$$0 < \frac{a}{2} < a_k \quad \text{eventually}\,.$$

Hence, for every $m \in \mathbb{R}_+$ we have

$$a_k \cdot b_k > \frac{a}{2} \cdot \frac{2m}{a} = m \quad \text{eventually}\,.$$

Therefore

$$\limsup_{k\to\infty} a_k b_k = \infty\,.$$

Finally, suppose that

$$\limsup_{k\to\infty} b_k = b \quad \text{for some } b \in \mathbb{R}\,.$$

We must show that

$$\limsup_{k\to\infty} a_k b_k = ab\,.$$

Let $\epsilon > 0$. Because $a > 0$, our supposition implies that

$$\begin{aligned} b_k &< b + \frac{\epsilon}{4a} \quad \text{eventually}\,,\\ b_k &> b - \frac{\epsilon}{4a} \quad \text{frequently}\,. \end{aligned}$$

Because $a_k \to a$ as $k \to \infty$ and $a > 0$ we know that

$$\begin{aligned} |a_k b - ab| &< \frac{\epsilon}{2} \quad \text{eventually}\,,\\ 0 < a_k &< 2a \quad \text{eventually}\,. \end{aligned}$$

Hence, for every $\epsilon > 0$ we have

$$\begin{aligned} a_k b_k &< a_k b + \frac{a_k}{2a}\frac{\epsilon}{2} < ab + \frac{\epsilon}{2} + \frac{\epsilon}{2} = ab + \epsilon \qquad \text{eventually}\,,\\ a_k b_k &> a_k b - \frac{a_k}{2a}\frac{\epsilon}{2} > ab - \frac{\epsilon}{2} - \frac{\epsilon}{2} = ab - \epsilon \qquad \text{frequently}\,. \end{aligned}$$

Therefore

$$\limsup_{k\to\infty} a_k b_k = ab\,.$$

$\square$

4. [15] Let $a_0 > 0$ and define $\{a_k\}_{k\in\mathbb{N}}$ by $a_{k+1} = \frac{2}{3}(a_k + 1/a_k^2)$ for every $k \in \mathbb{N}$.
   a. [10] Prove that $\{a_k\}_{k\in\mathbb{N}}$ converges.
   b. [5] Evaluate $\lim\limits_{k\to\infty} a_k$. Give your reasoning.

Solution (a). We will show that the sequence $\{a_k\}_{k\in\mathbb{N}}$ is contracting, whereby it will be convergent. Our first step is to bound a_k below for every $k \in \mathbb{Z}_+$. Let $a_0 > 0$. Because

$$a_{k+1} = \tfrac{2}{3}\left(a_k + \frac{1}{a_k^2}\right) \quad \text{for every } k \in \mathbb{N}\,,$$

it follows by induction that $a_k > 0$ for every $k \in \mathbb{N}$. We see that

$$a_{k+1} = \tfrac{2}{3} f(a_k) \quad \text{for every } k \in \mathbb{N}\,,$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$f(x) = x + \frac{1}{x^2} \quad \text{for every } x > 0\,.$$

Because

$$f'(x) = 1 - \frac{2}{x^3} \quad \text{and} \quad f''(x) = \frac{6}{x^4} > 0 \quad \text{for every } x > 0\,,$$

we see that $x = 2^{\frac{1}{3}}$ is the unique minimizer of $f(x)$ over $x > 0$, whereby

$$f(x) \geq f\left(2^{\frac{1}{3}}\right) = 2^{\frac{1}{3}} + \frac{1}{2^{\frac{2}{3}}} = \tfrac{3}{2}\, 2^{\frac{1}{3}} \quad \text{for every } x > 0.$$

Therefore

$$a_{k+1} = \tfrac{2}{3} f(a_k) \geq \tfrac{2}{3} f\left(2^{\frac{1}{3}}\right) = \tfrac{2}{3}\,\tfrac{3}{2}\, 2^{\frac{1}{3}} = 2^{\frac{1}{3}} \quad \text{for every } k \in \mathbb{N}\,,$$

whereby $a_k \geq 2^{\frac{1}{3}}$ for every $k \in \mathbb{Z}_+$.

Now we show that $\{a_k\}_{k\in\mathbb{N}}$ is a contracting sequence. For every $k \in \mathbb{Z}_+$ we have

$$\begin{aligned} a_{k+1} - a_k &= \tfrac{2}{3}\left(a_k + \frac{1}{a_k^2} - a_{k-1} - \frac{1}{a_{k-1}^2}\right) \\ &= \tfrac{2}{3}\left(1 - \frac{a_k + a_{k-1}}{a_k^2 a_{k-1}^2}\right)(a_k - a_{k-1})\,. \end{aligned}$$

Because $a_k \geq 2^{\frac{1}{3}}$ for every $k \in \mathbb{Z}_+$, we see that

$$0 < \frac{a_k + a_{k-1}}{a_k^2 a_{k-1}^2} = \frac{1}{a_k\, a_{k-1}^2} + \frac{1}{a_k^2 a_{k-1}} \leq \frac{1}{2} + \frac{1}{2} = 1 \quad \text{for every } k \geq 2\,.$$

Therefore

$$|a_{k+1} - a_k| \leq \tfrac{2}{3} |a_k - a_{k-1}| \quad \text{for every } k \geq 2\,,$$

whereby $\{a_k\}_{k\in\mathbb{N}}$ is a contracting sequence. Because every contracting sequence is convergent, $\{a_k\}_{k\in\mathbb{N}}$ converges. $\square$

Solution (b). Because $\{a_k\}_{k\in\mathbb{N}}$ converges, let

$$a = \lim_{k\to\infty} a_k \,.$$

Because $a_k > 2^{\frac{1}{3}}$ for every $k \geq 1$, we know that $a \geq 2^{\frac{1}{3}}$. Because $a > 0$ and

$$a_{k+1} = \tfrac{2}{3}\left(a_k + \frac{1}{a_k^2}\right) \quad \text{for every } k \in \mathbb{N}\,,$$

we have

$$a = \lim_{k\to\infty} a_{k+1} = \tfrac{2}{3}\lim_{k\to\infty}\left(a_k + \frac{1}{a_k^2}\right) = \tfrac{2}{3}\left(a + \frac{1}{a^2}\right) \,.$$

Upon solving for a we find that $a^3 = 2$, whereby $a = 2^{\frac{1}{3}}$. $\square$

5. [10] Let $A, B \subset \mathbb{R}$. Show that $(A \cup B)^c = A^c \cup B^c$.
(Here S^c denotes the closure of any $S \subset \mathbb{R}$.)

Solution. First, we show that $A^c \cup B^c \subset (A \cup B)^c$. Let $x \in A^c \cup B^c$. Then either $x \in A^c$ or $x \in B^c$. Without loss of generality we can suppose that $x \in A^c$. Then there exists a sequence $\{x_k\} \subset A$ such that $x_k \to x$ as $k \to \infty$. Because $A \subset A \cup B$, we have $\{x_k\} \subset A \cup B$ and $x_k \to x$ as $k \to \infty$, whereby $x \in (A \cup B)^c$. Therfore $A^c \cup B^c \subset (A \cup B)^c$.

Next, we show that $(A \cup B)^c \subset A^c \cup B^c$. Let $x \in (A \cup B)^c$. Then there exists a sequence $\{x_k\} \subset A \cup B$ such that $x_k \to x$ as $k \to \infty$. Then either

$$x_k \in A \text{ frequently} \quad \text{or} \quad x_k \in B \text{ frequently}\,,$$

because otherwise we would have

$$x_k \notin A \text{ eventually} \quad \text{and} \quad x_k \notin B \text{ eventually}\,,$$

which implies that

$$x_k \notin A \cup B \text{ eventually}\,,$$

which contradicts the fact that $\{x_k\} \subset A \cup B$. Without loss of generality we can suppose that $x_k \in A$ frequently, whereby there exists a subsequence $\{x_{k_n}\} \subset A$. Because $\{x_{k_n}\} \subset A$ and $x_{k_n} \to x$ as $n \to \infty$, we conclude that $x \in A^c$, whereby $x \in A^c \cup B^c$. Therefore $(A \cup B)^c \subset A^c \cup B^c$. $\square$

6. [10] Determine all $a \in \mathbb{R}$ for which
$$\sum_{k=0}^{\infty} \left(\frac{\sqrt{k^2+1}}{k^3+4} \right)^a \quad \text{converges}\,.$$
Give your reasoning.

Solution. We will compare this series to the p-series
$$\sum_{k=1}^{\infty} \frac{1}{k^{2a}}\,,$$
which converges when $2a > 1$ and diverges otherwise. We will use the Two-Way Limit Comparison Test. Because
$$\lim_{k\to\infty} \left(k^2 \frac{\sqrt{k^2+1}}{k^3+4} \right) = \lim_{k\to\infty} \frac{\sqrt{1+\dfrac{1}{k^2}}}{1+\dfrac{4}{k^3}} = 1\,,$$
we see that
$$\lim_{k\to\infty} \frac{\left(\dfrac{\sqrt{k^2+1}}{k^3+4} \right)^a}{\dfrac{1}{k^{2a}}} = \lim_{k\to\infty} \left(k^2 \frac{\sqrt{k^2+1}}{k^3+4} \right)^a = 1\,.$$
The Two-Way Limit Comparison Test then says that the two series are either both convergent or both divergent. Because the p-series converges when $2a > 1$ and diverges otherwise, the set of $a \in \mathbb{R}$ for which the given series converges is $(\frac{1}{2}, \infty)$. $\square$

7. [15] Determine the set $C \subset \mathbb{R}$ defined by
$$C = \left\{ x \in \mathbb{R} \,:\, \sum_{k=1}^{\infty} \frac{x^k}{k} \quad \text{converges} \right\}.$$
Give your reasoning. (The set is an interval. Be sure to check its endpoints!)

Solution. Because
$$\lim_{k\to\infty} \frac{\left| \dfrac{x^{k+1}}{k+1} \right|}{\left| \dfrac{x^k}{k} \right|} = |x| \lim_{k\to\infty} \frac{k}{k+1} = |x|\,,$$
the Ratio Test tells us that the series

- converges absolutely when $|x| < 1$ and
- diverges when $|x| > 1$.

However it says nothing when $x = \pm 1$, so these cases must be check separately.

When $x = 1$ the series becomes the harmonic seies
$$\sum_{k=1}^{\infty} \frac{1}{k}\,,$$
which is divergent.

When $x = -1$ the series becomes the alternating harmonic seies

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k},$$

which is convegent.

Therefore $C = [-1, 0)$. □

8. [15] Use facts from calculus to prove the following.

a. [5] Show for every $x < 1$ and every $n \in \mathbb{Z}_+$ that

$$\sum_{k=1}^{n} \frac{x^k}{k} = \log\left(\frac{1}{1-x}\right) - \int_0^x \frac{y^n}{1-y}\,\mathrm{d}y\,.$$

b. [5] Show for every $n \in \mathbb{Z}_+$ that

$$\left|\int_0^x \frac{y^n}{1-y}\,\mathrm{d}y\right| \leq \begin{cases} \dfrac{1}{n+1}\dfrac{x^{n+1}}{1-x} & \text{for } x \in (0,1)\,, \\[2ex] \dfrac{1}{n+1}|x|^{n+1} & \text{for } x \leq 0\,. \end{cases}$$

c. [5] Let $C \subset \mathbb{R}$ be defined as in Problem 7. Show for every $x \in C$ that

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = \log\left(\frac{1}{1-x}\right).$$

Solution (a). Let $n \in \mathbb{Z}_+$. The difference of powers formula gives

$$1 - y^n = (1-y)\sum_{k=0}^{n-1} y^k\,,$$

whwereby

$$\sum_{k=0}^{n-1} y^k = \frac{1-y^n}{1-y}\,, \qquad \text{for every } y \neq 1\,.$$

Upon integrating this equation we obtain

$$\int_0^x \sum_{k=0}^{n-1} y^k\,\mathrm{d}y = \int_0^x \frac{1}{1-y}\,\mathrm{d}y - \int_0^x \frac{y^n}{1-y}\,\mathrm{d}y\,, \quad \text{for every } x < 1\,,$$

Because

$$\int_0^x \sum_{k=0}^{n-1} y^k\,\mathrm{d}y = \sum_{k=0}^{n-1}\int_0^x y^k\,\mathrm{d}y = \sum_{k=0}^{n-1}\frac{x^{k+1}}{k+1} = \sum_{k=1}^{n}\frac{x^k}{k}\,, \quad \text{for every } x \in \mathbb{R}\,,$$

while

$$\int_0^x \frac{1}{1-y}\,\mathrm{d}y = -\log(1-y)\Big|_o^x = -\log(1-x) = \log\left(\frac{1}{1-x}\right), \quad \text{for every } x < 1\,,$$

we conclude that

$$\sum_{k=1}^{n} \frac{x^k}{k} = \log\left(\frac{1}{1-x}\right) - \int_0^x \frac{y^n}{1-y}\,\mathrm{d}y\,, \quad \text{for every } x < 1\,,$$

Solution (b). If $x \in (0,1)$ then

$$0 < \int_0^x \frac{y^n}{1-y}\,\mathrm{d}y < \int_0^x \frac{y^n}{1-x}\,\mathrm{d}y = \frac{1}{n+1}\,\frac{x^{n+1}}{1-x}\,.$$

If $x \leq 0$ then by the change of variables $z = -y$ we see that

$$\left|\int_0^x \frac{y^n}{1-y}\,\mathrm{d}y\right| = \int_0^{|x|} \frac{z^n}{1+z}\,\mathrm{d}z \leq \int_0^{|x|} z^n\,\mathrm{d}z = \frac{1}{n+1}|x|^{n+1}\,.$$

Therefore

$$\left|\int_0^x \frac{y^n}{1-y}\,\mathrm{d}y\right| \leq \begin{cases} \dfrac{1}{n+1}\dfrac{x^{n+1}}{1-x} & \text{for } x \in (0,1)\,, \\[2ex] \dfrac{1}{n+1}|x|^{n+1} & \text{for } x \leq 0\,. \end{cases}$$

Solution (c). If $x \in (0,1)$ then

$$\limsup_{n\to\infty} \left|\int_0^x \frac{y^n}{1-y}\,\mathrm{d}y\right| \leq \limsup_{n\to\infty} \frac{1}{n+1}\frac{x^{n+1}}{1-x} = 0\,.$$

If $x \in [-1,0]$ then

$$\limsup_{n\to\infty} \left|\int_0^x \frac{y^n}{1-y}\,\mathrm{d}y\right| \leq \limsup_{n\to\infty} \frac{1}{n+1}|x|^{n+1} = 0\,.$$

Hence, for every $x \in [-1,1)$ we have

$$\lim_{n\to\infty} \left|\int_0^x \frac{y^n}{1-y}\,\mathrm{d}y\right| = 0\,.$$

Because for every $n \in \mathbb{Z}_+$ we have

$$\left|\sum_{k=1}^{n} \frac{x^k}{k} - \log\left(\frac{1}{1-x}\right)\right| \leq \left|\int_0^x \frac{y^n}{1-y}\,\mathrm{d}y\right|\,,$$

it follows that for every $x \in [-1,1)$ we have

$$\lim_{n\to\infty} \left|\sum_{k=1}^{n} \frac{x^k}{k} - \log\left(\frac{1}{1-x}\right)\right| = 0\,,$$

whereby

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = \log\left(\frac{1}{1-x}\right)\,, \quad \text{for every } x \in [-1,1)\,.$$