Second Exam
Math 410, Professor David Levermore
due 4:00pm Friday, 6 November 2020

Indicate your answer to each part of each question clearly. Work that you do not want considered should be crossed out. Your reasoning must be given for full credit. Good luck!

1. [10] Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$
Show that f maps intervals to intervals yet is not continuous over $\mathbb{R}$.

2. [15] Let $D \subset \mathbb{R}$ be unbounded above. Let $f : D \to \mathbb{R}$. Let $b \in \mathbb{R}$. Prove that
$$\lim_{x\to\infty} f(x) = b$$
if and only if for every $\{x_n\} \subset D$
$$\lim_{n\to\infty} x_n = \infty \qquad \Longrightarrow \qquad \lim_{n\to\infty} f(x_n) = b\,.$$

3. [10] Show that there is a function $f : [-1, 1] \to \mathbb{R}$ that is differentiable over $[-1, 1]$ but whose derivative f' is not continuous over $[-1, 1]$.

4. [15] Prove for every nonzero $x \in \mathbb{R}$ that
$$1 + \tfrac{8}{5}x < (1 + x)^{\frac{8}{5}}\,.$$

5. [10] Prove that $f(x) = e^{-2x}\sin(4x)$ is Lipschitz continuous over $[0, \infty)$ and find its smallest possible Lipschitz constant.

6. [15] Suppose that we are using the Newton-Raphson method to solve $x^3 - 6 = 0$. Use Proposition 7.11 in the notes to bound the error when our initial guess is 2.

7. [10] Show that the function $f(x) = \sin(1/x)$ is not uniformly continuous over $\mathbb{R}+$.

8. [15] Let $\alpha \in (0, 1)$. Prove that
$$|y^\alpha - x^\alpha| \leq |y - x|^\alpha \quad \text{for every } x, y \in [0, \infty)\,.$$
Hint: For every $x \in \mathbb{R}+$ define the function $g : [x, \infty) \to \mathbb{R}$ by
$$g(y) = (y - x)^\alpha - y^\alpha + x^\alpha \text{for every } y \in [x, \infty)\,,$$
and show that it is increasing over $[x, \infty)$.