Final Exam
Math 410, Professor David Levermore
due 4:00pm Friday, 18 December 2020

Indicate your answer to each part of each question clearly. Work that you do not want considered should be crossed out. Your reasoning must be given for full credit. Good luck!

(1) [15] Let $\{a_k\}$ be a montonic sequence in $\mathbb{R}$. Show that it converges if it has a convergent subsequence.

(2) [20] Let $\alpha \in (0,1)$ and $x > 0$. Define $g : [x, \infty) \to \mathbb{R}$ by

$$g(y) = (y-x)^\alpha - y^\alpha + x^\alpha \qquad \text{for every } y \in [x, \infty)\,.$$

Prove that g is increasing over $[x, \infty)$.

(3) [15] Prove that a countable union of sets, each of which has measure zero, also has measure zero.

(4) [50] Let $q \in \mathbb{R} - \mathbb{N}$. Let $f(x) = (1+x)^q$ for every $x > -1$. Then

$$f^{(k)}(x) = q(q-1)\cdots(q-k+1)(1+x)^{q-k} \;\text{ for every } x > -1 \text{ and } k \in \mathbb{Z}_+\,.$$

The formal Taylor series of f about 0 is therefore

$$1 + \sum_{k=1}^{\infty} \frac{q(q-1)\cdots(q-k+1)}{k!}\, x^k\,.$$

(a) [10] Show that this series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

(b) [15] Show that when $|x| < 1$ we have

$$1 + \sum_{k=1}^{\infty} \frac{q(q-1)\cdots(q-k+1)}{k!}\, x^k = (1+x)^q\,.$$

(c) [15] Show that when $q > -1$ we have

$$1 + \sum_{k=1}^{\infty} \frac{q(q-1)\cdots(q-k+1)}{k!} = 2^q\,,$$

while when $q \le -1$ the above series diverges.

(d) [10] Let $r \in (0,1)$. Show that when $q > -1$

$$\sum_{k=1}^{\infty} \frac{q(q-1)\cdots(q-k+1)}{k!}\, x^k \qquad \text{converges uniformly over } [-r, 1].$$

(5) [15] Let $[a,b] \subset \mathbb{R}$ be a closed, bounded interval. Let $f : [a,b] \to [a,b]$. Suppose there exists an $K \in (0,1)$ such that
$$|f(x) - f(y)| \leq K\,|x - y| \quad \text{for every } x, y \in [a,b]\,.$$
Let $x_0 \in [a,b]$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ by
$$x_{n+1} = f(x_n) \quad \text{for every } n \in \mathbb{N}\,.$$
Prove that $\{x_n\}_{n=0}^{\infty}$ is a contracting sequence.

(6) [15] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose that f' is decreasing over a bounded interval (a,b). Prove that f is strictly concave over $[a,b]$.

(7) [25] Let $g : \mathbb{R} \to \mathbb{R}$ be continuous over $\mathbb{R}$ such that
$$\lim_{x \to -\infty} g(x) = 0\,, \qquad \lim_{x \to \infty} g(x) = 0\,.$$

(a) [10] Prove that $g : \mathbb{R} \to \mathbb{R}$ is uniformly continuous over $\mathbb{R}$.

(b) [15] Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable over $[0,1]$. Define $h : \mathbb{R} \to \mathbb{R}$ by
$$h(x) = \int_0^1 g(x - z) f(z)\, \mathrm{d}z\,.$$
Prove that $h : \mathbb{R} \to \mathbb{R}$ is uniformly continuous over $\mathbb{R}$.

(8) [45] For every $n \in \mathbb{Z}_+$ define $h_n(x) = n^2 x e^{-nx}$ for every $x \in [0, \infty)$.

(a) [10] Prove that $h_n \to 0$ pointwise over $[0, \infty)$.

(b) [5] Prove that this limit is not uniform over $[0, \infty)$.

(c) [10] Prove that this limit is uniform over $[\delta, \infty)$ for every $\delta > 0$.

(d) [5] Prove for every $\delta > 0$ that
$$\lim_{n \to \infty} \int_0^{\delta} h_n = 1\,.$$

(e) [15] Let $g : [0,1] \to \mathbb{R}$ be continuous. Show that
$$\lim_{n \to \infty} \int_0^1 h_n\, g = g(0)\,.$$