Fifth Homework: MATH 410
Due Friday, 2 October 2020

1. Let $\{a_k\}_{k\in\mathbb{N}}$ be a real sequence and $\{a_{n_k}\}$ be any subsequence. Show that
$$\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely}\,.$$
2. Prove Proposition 3.14 in the notes.
3. Give examples of both a divergent series and a convergent series such that
$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = 1\,.$$
4. Consider the set
$$\left\{ x \in \mathbb{R} \,:\, \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!}\,\frac{n!}{(3n)!}\,x^n \quad \text{converges} \right\}.$$
Use the root test to prove that this set is an interval and find its endpoints. You may use the fact that
$$\lim_{k\to\infty} \frac{\sqrt[k]{k!}}{k} = \frac{1}{e}\,.$$
5. Show that if neither condition of Proposition 3.15 is satisfied then the series may either converge or diverge.
6. Consider the set
$$\left\{ x \in \mathbb{R} \,:\, \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!}\,\frac{n!}{(3n)!}\,x^n \quad \text{converges} \right\}.$$
Use the ratio test to prove that this set is an interval and find its endpoints.
7. Show that if neither condition of Proposition 3.17 is satisfied then the series may either converge or diverge.
8. Determine all $x, p \in \mathbb{R}$ for which the Fourier p-series
$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p} \quad \text{converges}\,.$$
9. Prove Proposition 4.1 in the notes.
10. Prove Proposition 4.2 in the notes.
11. Prove Proposition 4.6 in the notes.
12. Prove Proposition 4.8 in the notes.